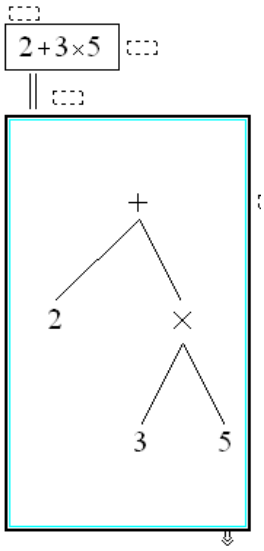
Exploring the structure of numerical expressions

Scenario Identity



- **Writers:**
  Chiappini G., Pedemonte B., Robotti E.
- **Subject area:**

  Arithmetic
- **Topic(s):**

  Understanding the structure of numerical expressions

Activity rationale

- **Teaching and Learning problem addressed**

Research highlights obstacles students have in treating expressions. For example, students tend to interpret expressions such as 3+4k as 7k. This obstacle arises because of the way we read from left to right (Tall and Thomas, 1991). As a matter of fact, from prior experience with arithmetic students expect to perform some calculation to produce a result. 3+4k could seem an incomplete result. In a sense students do not see expressions as having a dual nature: that of process and of a product. The expression 3+4k indicates both the instructions to perform a calculation (process) and the result of a such calculation when a value is not assigned to the variable (product). Obstacles mainly arise because students usually do not see expression as a process but only as a product. As a consequence, many students are unable to judge the equivalence of expressions like 685-492+947 and 947-492+685 without recourse to computation (Chaiklin and Lesgold, 1984). Research in the educational field highlights that students' competencies of the numerical expressions are mainly computational: students usually know the priority of the operations and the use of parentheses and are able to use them in procedural way to calculate the results of simple numeric expressions. Whether they have to convert an expression in verbal language, they usually realize a stenographic translation of its linear representation. Kieran (1989) highlights that an important aspect of the students' difficulties in manipulating expressions is to recognise and use *structure.* Structure includes the "surface" structure and the systemic structure. Mistakes emerging in the manipulation of expressions clearly evidence students' difficulty to control the expression on structural level. Students can manage the "superficial" structure (Kieran) of expressions, but they are not always able to manage their "systemic" structure (Kieran).

This scenario is aimed to support the comprehension of the structure of numerical expressions. To pursue this didactical objective, this scenario is centred on the integrated use of Aplusix DDA and in particular on its models for tree representations.

This scenario is based on two theoretical frameworks: the Semiotic registers of representation (R. Duval) and the Activity theory frame (Cole & Engeström, 1991). The hypothesis based on these frames is that specific tasks related to an integrated use of Aplusix, could favour students in understanding the structure of numerical expressions. As a matter of fact the ability to represent a given mathematical concept in at least

two registers and to perform conversions from one register to another should be an indicator of conceptual understanding of the notion. As a matter of fact, the theory of the Semiotic registers of representation (Duval, 1993) states that the ability to represent a given mathematical concept in at least two registers and to perform conversions from one register to another could be an indicator of conceptual understanding of a notion. In this scenario three registers of representation for an expression are considered: linear representation, tree representation and natural language.

Innovation

The innovative idea of this scenario is exploiting the new representative and operative potential of Aplusix to provide different representation systems (linear representation, tree representation and verbal representation) for understanding the structure of expression.

- **Added value**

Aplusix can be exploited to learn the tree representation of linear expression, to learn to convert a linear expression (expressed in symbols or in natural language) in a tree representation, and to learn to convert a tree representation in linear expression (expressed in symbols or in natural language). The main feature of Aplusix used to pursue these educational goals is the feedback that is available with this artefact to verify the equivalence between two linear expressions, between two tree representations or between linear expression and tree representation.

Moreover, two specific characteristics of the system can be used to edit expressions as trees: the Controlled Tree representation and the Mixed Tree representation.

When a tree is edited in the Controlled Tree representation there are some constraints and scaffolding: internal nodes must be operators and leaves must be numbers or variables.

When a tree is edited in the Mixed representation there are less constraints: each leaf of the tree can also be an usual representation. The usual representation can be expanded as a tree by clicking at the "+" button that appears when the mouse cursor is near a node; a tree, or a part of a tree, can be collapsed into a usual representation by clicking at the "-" button that appears when the mouse cursor is near a node.

The Controlled tree representation can be used at the beginning of the experiment to teach students to build a tree representation.

Then the Mixed tree representation could be more appropriate then the Controlled one because students is free to insert in leaves not only numbers but also expressions. As a matter of fact, the Mixed tree representation can be used as a validation tool: working in paper and pencil, students is asked to construct tree representation by linear expression or linear expression by tree representation. They can verify their answers using the Mixed representation mode of Aplusix inserting the expression and building the tree representation. They can compare if the tree representation of the screen and the tree representation performed with paper and pen are coincident or not.

Context of implementation

- **Educational Goals:**

The didactical goal of this activity concerns understanding the structure of numerical expressions. In particular:

- to learn the tree representation of linear expression,
- to learn to convert a linear expression (expressed in symbols or in natural language) in a tree representation,

- to learn to convert a tree representation in linear expression (expressed in symbols or in natural language).conditioned equality.

- **Population:**

  7th grade Students.

- **Students' prerequisites**:

  Knowledge of the operations with numbers (integers and rational numbers)
  Basic skills in solving numerical expressions
  Familiarity with basic computer functions

- **Duration**:

  10 school hours.

- **Place**:

  The school computer lab.

- **Resources and tools:**

  This scenario is completely based on the use of Aplusix.

  Aplusix is an application for helping secondary school students to learn algebra. It lets students solve exercises and provides feedback: it verifies the correctness of the calculations and of the end of the exercises.

  Aplusix has been designed to be integrated into the regular work of the class: it is close to the paper-pencil environment, it uses a very intuitive editor of algebraic expressions (in two dimensions); it contains 400 patterns of exercises organized by themes (numerical calculation, expansion, factorization, and solving equations, inequations and systems of equations) and by complexity. It also contains an exercise editor allowing teachers to build their own lists of exercises.

  The application records all of the students' actions. This allows the student and the teacher to observe them later with a "Replay system". Teachers also have access to statistics concerning their classes indicating the amounts of exercises they worked on, amounts of well-solved exercises, amounts of incorrect calculations, and scores.

  Aplusix runs on the local network of the school. An administration application allows managing classes, teachers and students (account creation, modification and suppression). Aplusix can also be installed on a personal computer in particular at home.

  Tree representation mode

  Aplusix-Tree allows the use of tree representations of algebraic expressions. It also includes two new types of exercises: "*Transform a usual (symbolic) representation into a tree representation*" and "*Transform a tree representation into a usual representation*".

  There are four types of representation:

  - Usual representation: the "standard" (symbolic) representation of algebraic expressions.
  - Free tree representation: expressions can be edited as trees. In this mode, there is no constraint and no verification of the tree when it is edited (all sort of incorrect trees can be built).

- Controlled tree representation: there are constraints and scaffolding when a tree is edited: internal nodes must be operators and leaves must be numbers or variables. The arity of the operators must be correct.
- Mixed representation: each leaf of the tree is a usual representation of an expression. A usual representation can be expanded as a tree by clicking at the "+" button that appears when the mouse cursor is near a node; a tree, or a part of a tree, can be collapsed into a usual representation by clicking at the "-" button that appears when the mouse cursor is near a node.

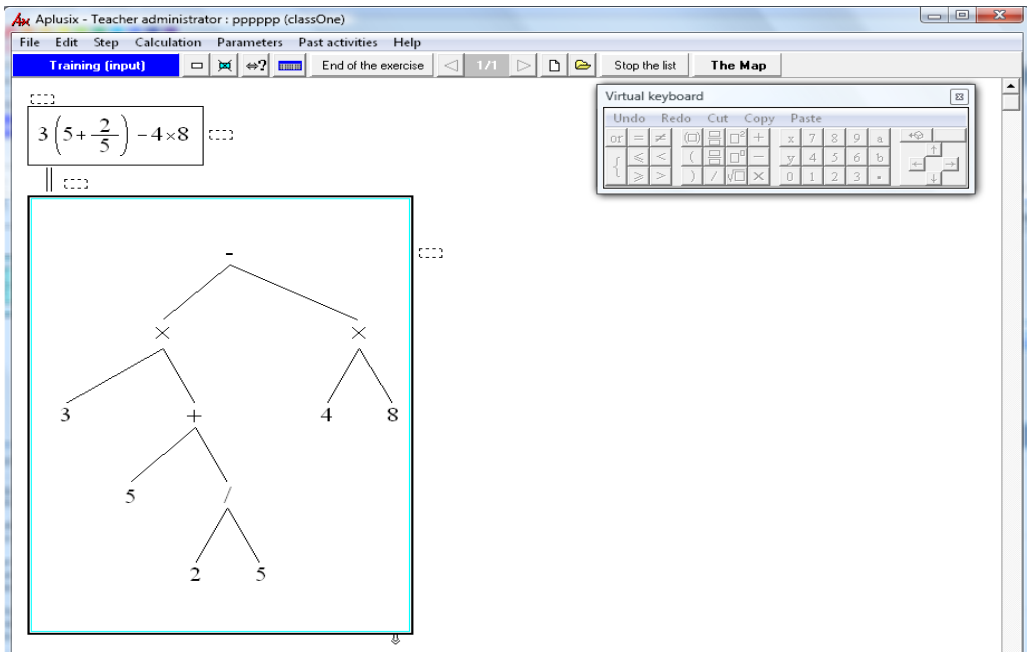


Fig. 2 The expression 3(5+2/5)-4*8 has been inserted in Aplusix and a tree representation of it has been constructed

- **Types of activities**

In this scenario, Aplusix is used to compare and to validate linear representation and tree representation for expression. The modalities of employment of this tool inside the scenario is based on the Activity theory frame. According to it, learning can emerge overcoming contradictions that can appear during educational activities. Tasks of this scenario have been designed to be source of contradiction through a comparison of the pen and paper solution and the solution performed in Aplusix.

Enactment of the activity

- **Activities**

This scenario comprises 4 distinct sections:

1. Card 1: Introduction to the tree construction in Aplusix
2. Card 2: Relationship between a linear representation and a tree representation of expressions
3. Card 3:How many tree representations can be found for a linear expression?
4. Card 4: Relationship among linear representation, tree representation and natural language representation of expression

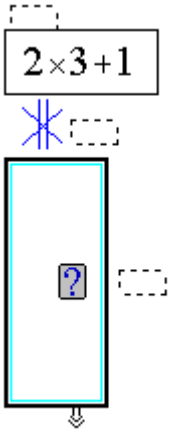
Card 1. Introduction to the tree construction in Aplusix

1) Write the expression (2*3+1)**+**(1/2*3). Duplicate the cell (clicking on the arrow under the cell). Visualise step by step the construction of the expression tree representation by means of the function “Mixed representation”. You can select this modality by the representation menu visualised with the mouse right button.

Write the expression (2*3+1)/5. Duplicate the cell (clicking on the arrow under the cell). Visualise step by step the construction of

the expression tree representation by means of the function “Mixed representation”.

2) Write the expression 2*3+1. By the mouse right button select “Step_New Step”. A empty cell appears.

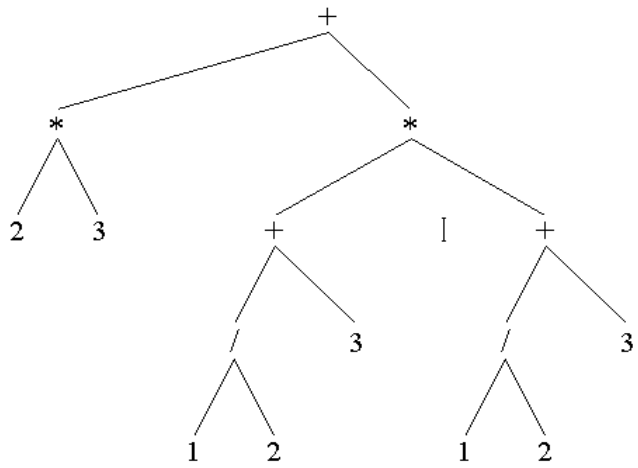


Construct the tree after selecting “Mixed_Representation”. You can expand the representation using the button +.

3) Repeat the previous exercise with the expression (2*3+1) /5. Insert a new step to the expression and construct the tree representation using the Controlled Representation mode. Construct the tree representation to the expression (2*3+1)2 using “Controlled Representation” mode.

4) *The following tree representation is provided to students in Aplusix by the Mixed Representation mode.*

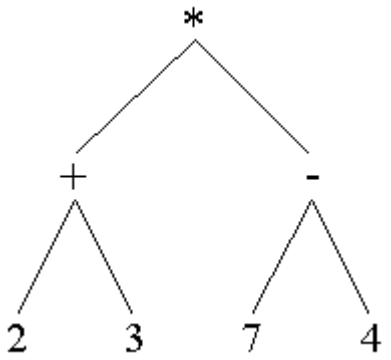
Write in your notebook the linear expression corresponding to the tree representation.



Compress node by node the tree representation to visualise the linear expression. The obtained expression is equal to the expression written in your notebook? If this is not the case, why they aredifferent?

5) *The following tree representation is provided to students in Aplusix by the Mixed Representation mode.*

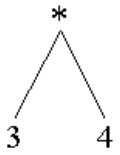
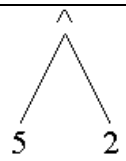
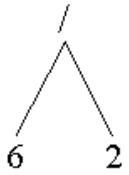
Write in your notebook the linear expression corresponding to the tree representation.

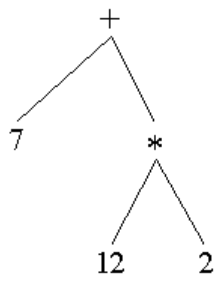
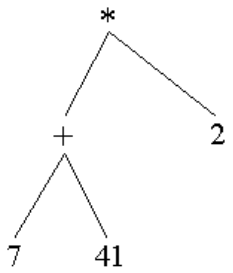
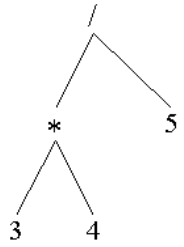
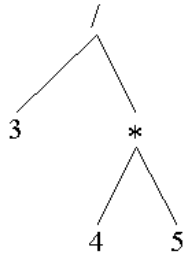
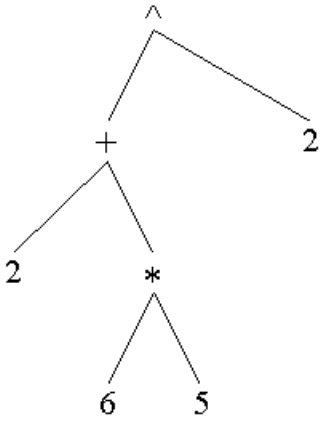


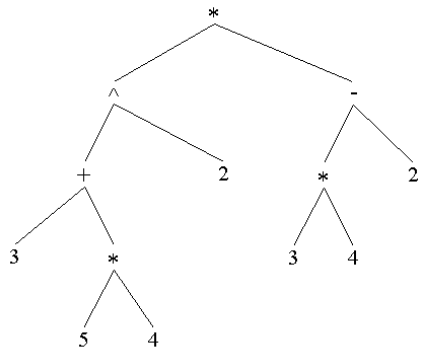
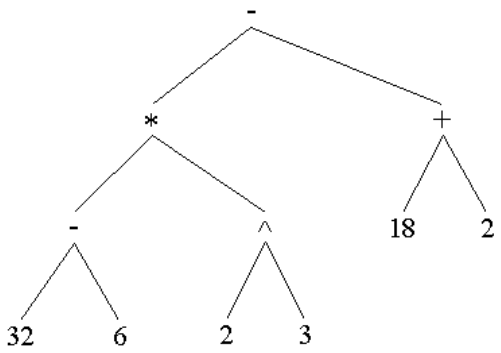
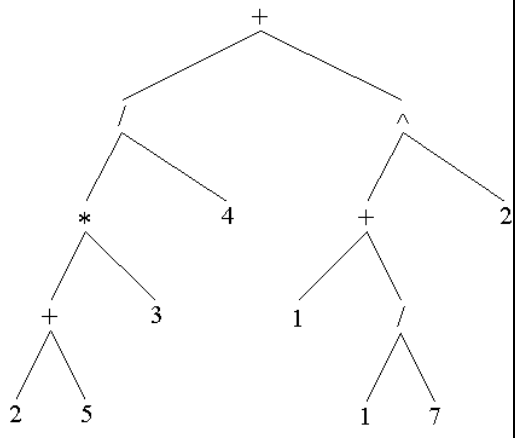
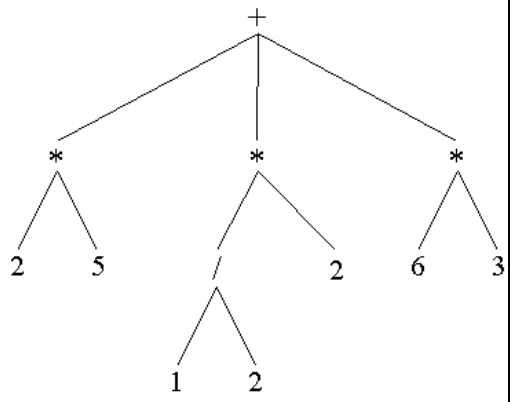
Compress node by node the tree representation to visualise the linear expression. The obtained expression is equal to the expression written in your notebook? If this is not the case, why they aredifferent?

Card 2: Relationship between a linear representation and a tree representation of expressions

1) In the right space of the table write the linear expression of each tree representations

1.		
2.		
3.		
4.		

5.	 + 7 * 12 2	
6.	 * + 2 7 41	
7.	 / * 5 3 4	
8.	 / 3 * 4 5	
9.	 ^ + 2 2 * 6 5	

10.	 * ^ - + 2 * 2 3 * 3 4 5 4	
11.	 - * + - ^ 18 2 32 6 2 3	
12.	 + / ^ * 4 + 2 + 3 1 / 2 5 1 7	
13.	 + * * * 2 5 / 2 6 3 1 2	

2) Verify if each linear expression, written in the table, corresponds to the tree representation, writing in Aplusix each linear expression and transforming it in a tree. In Aplusix use the feedback provided by the system to see, step by step, if your construction is correct.
Is the tree representation in Aplusix the same tree representation that you find in the table?

3) For each linear expression construct the tree representation :

1.	$3+4\times5$	
2.	$3\times4+5$	
3.	$3\times(4+5)$	
4.	$3+(4\times5)$	
5.	$\frac{2}{5}+\frac{3}{2}+\frac{1}{7}$	
6.	$\frac{2+5}{3+2}$	
7.	$\frac{2+5}{6}\times\frac{2+1}{7}$	
8.	$5\times2-6\times3-\frac{1}{2}\times5$	

For each espression:

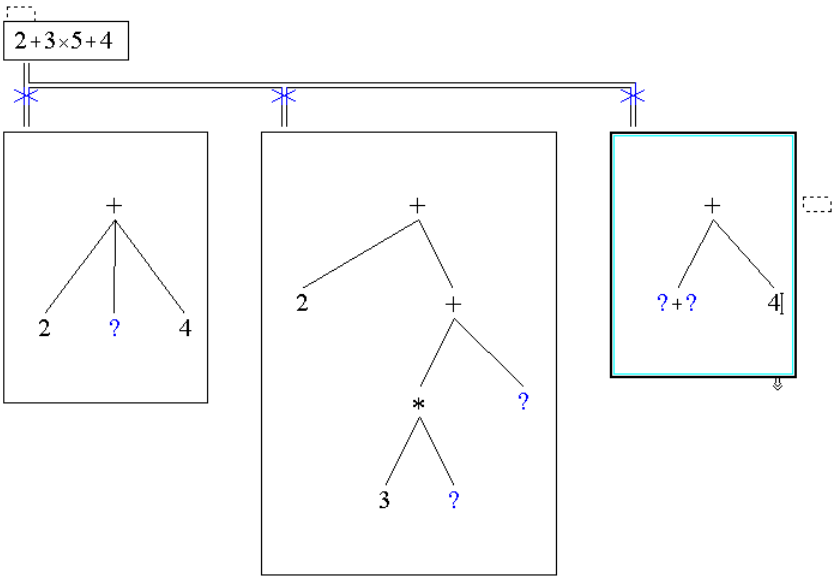
- Construct in Aplusix the found tree
- Verify if the two trees are equal

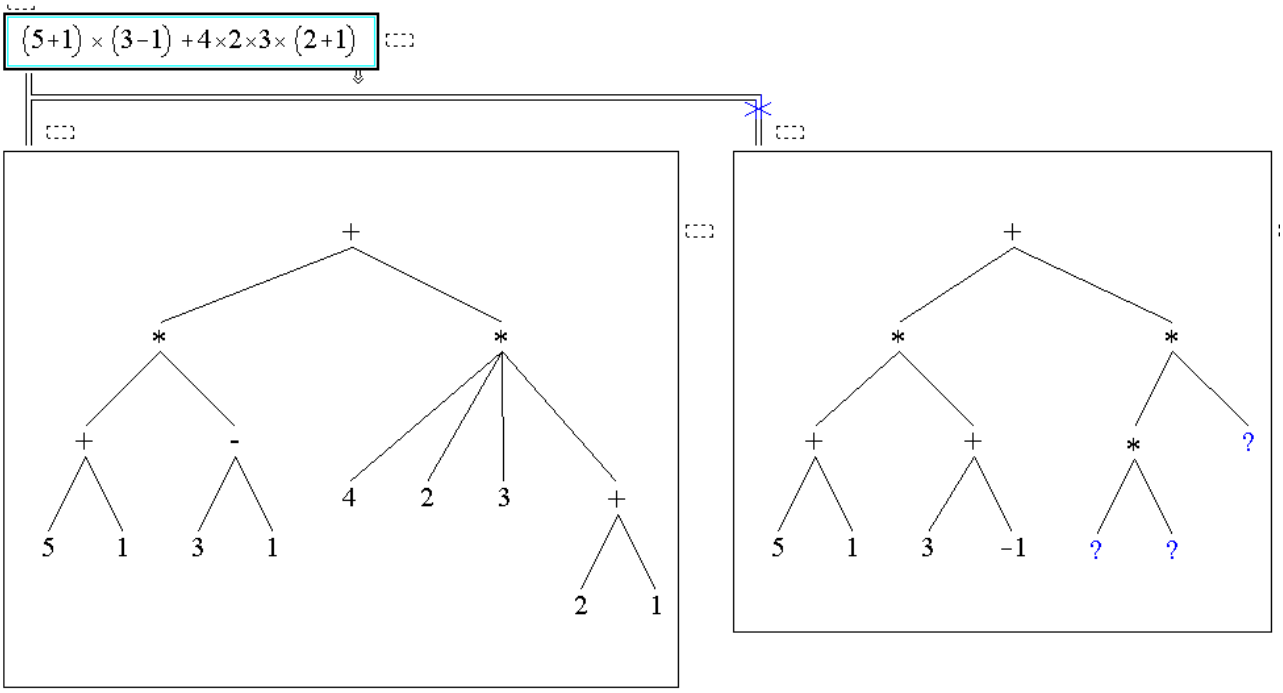
In the tree representations constructed in Aplusix, the operation situated at the top of the representation is always the same? When is the operation situated at the top of the tree representation performed?

Card 3: How many tree representations can be found for a linear expression?

1) Consider the following expression 12+2+15+(3*5)
Construct the corresponding tree representation. Is this the only tree representation of this expression or is it possible to construct other representations of this expression? Try to justify your answer.
Is there any case in which the tree representation is the only tree representation? Construct some examples.

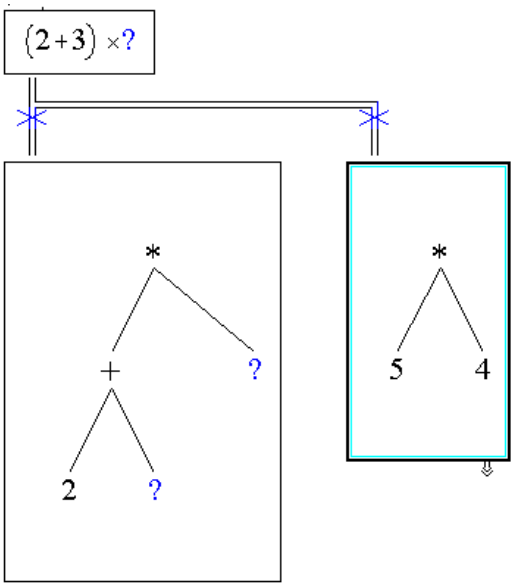
2) Complete the following tree representations and verify in Aplusix the given answers.





3) Given a tree is it possible to have more linear expressions of it ? Justify your answer

4) In the following figures, replace numbers to the question marks. (The question marks can be replaced by different numbers and not necessary by the same number). Using Aplusix verify your answers.


(2+3) ×?
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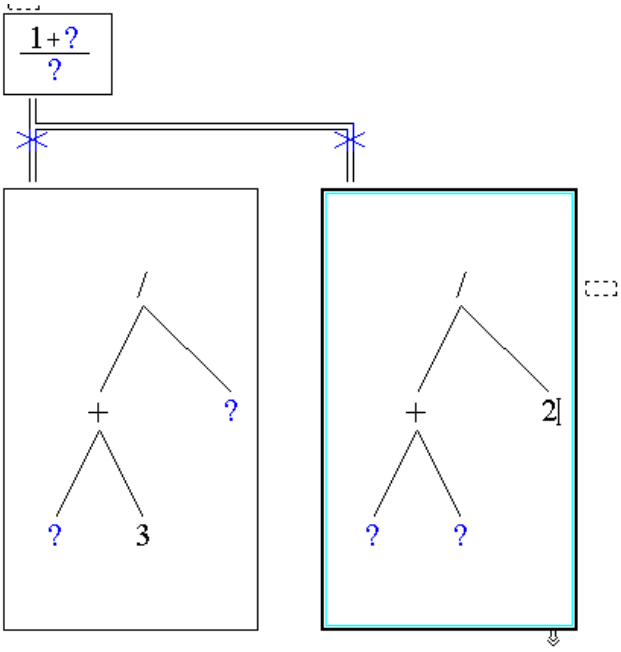
5
4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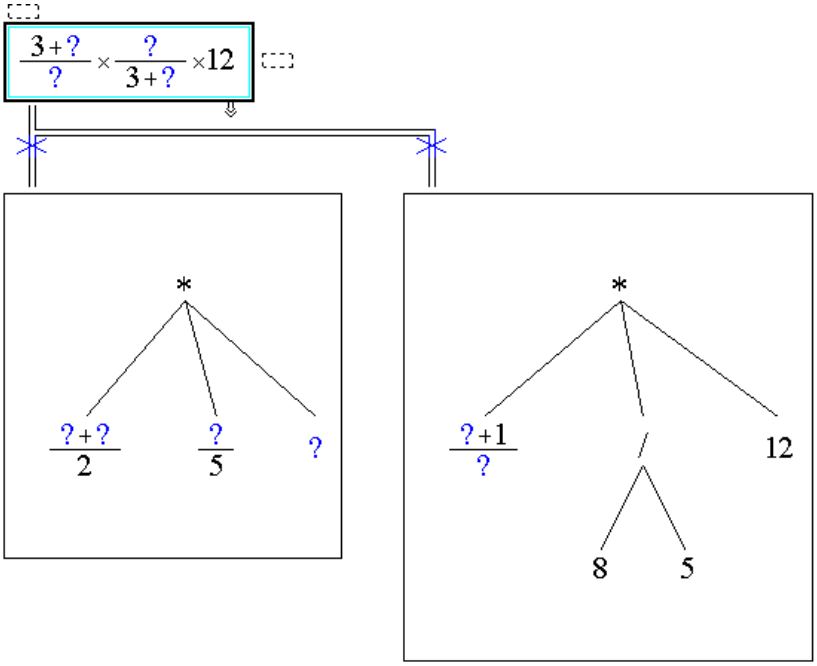
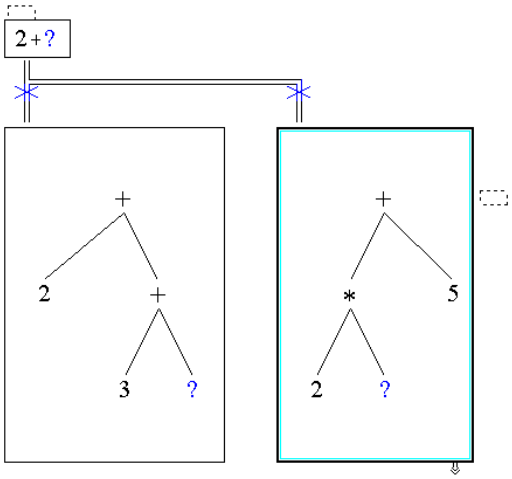
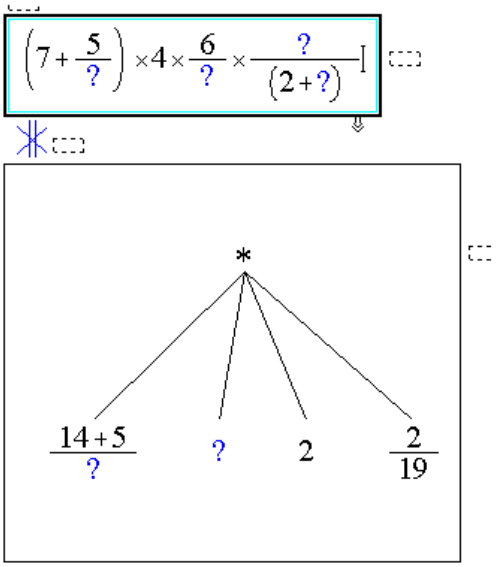






Card 4: Relationship among linear representation, tree representation and natural language representation of expression

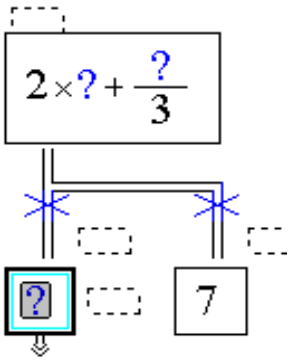
1) Construct a linear representation and a tree representation for each of the following statement:

Addition between 2 and 7		
Subtraction between 53 and 35		
3 times 4 and the addition between 5 and 2		
Subtraction between 15 and the division between 9 and 3		
Division between 6 and the division between 4 and 2		
Subtraction between the addition between 14 and 23, and the square of 5		
Square of the addition between 5 and 1 and the division between 5 and 7		
3 times 2 multiplied the addition between 4 and 2		

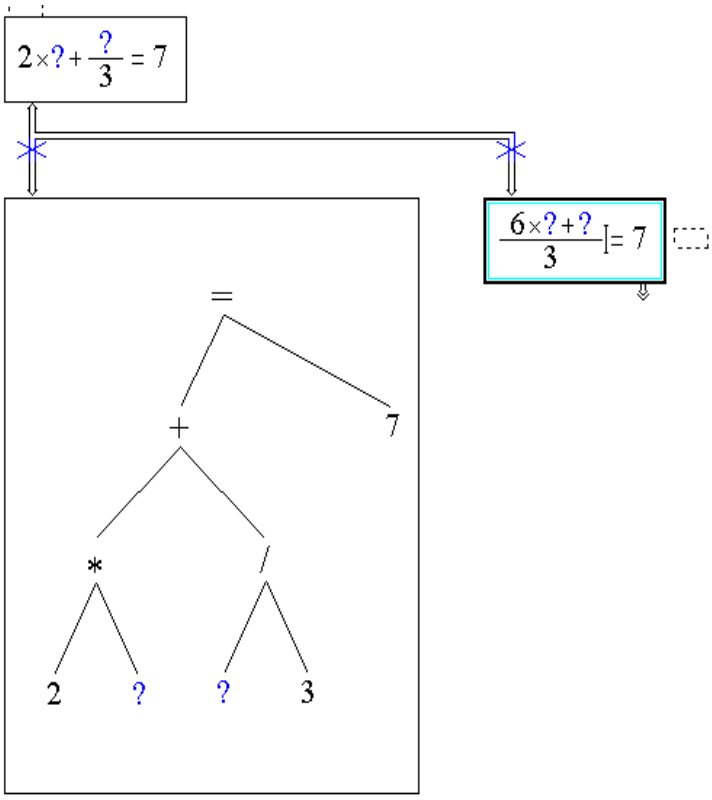
2)

a. Write as linear expression the following statement: multiply the double of a quantity added to 1 with the half of this quantity.
b. Write as tree representation the following statement: multiply the double of a quantity added to 1 with the half of this quantity.
In Aplusix, verify that the tree representation constructed in b corresponds to the tree representation that it is possible to construct by the linear expression found in a.

3) Consider the following statement: “the addition between the double of a number and the third part of this number is 7”. Which is this number? In Aplusix replace the question marks with this number. In this case, the question mark in each figure has to be replaced with the same number)



Or



4) Consider the subtraction between a number and 5/3. If the result is 1/3 what is this number?

- **Resources and tools**
  Worksheet, Aplusix

- **Organization of classroom activity**

  Students work in pairs. Each pair works on a computer with Aplusix and produces only one protocol containing the answers to the activity.

- **Assessment Suggestions**

  We describe a possible way a teacher can manage the activities presented in the described cards.

Card 1

This card presents tasks to introduce students to the tree representation of a numerical expression making use of two specific functionnalities of Aplusix: the mixed tree representation and the controlled tree representation.
At the beginning the teacher gives some information about Aplusix tree representation. In particular he/she explains that there are two ways to construct a tree: the mixed representation and the controlled representation modes. The teacher explains that the focus of the lesson is to understand how constructing a tree from a linear representation of expression.
At the end of the activity a brief discussion could be proposed by the teacher to institusionalise the use of the two specific function of Aplusix which allow to construct the tree representation.

Card 2

The tasks proposed in this card concern the relationship between a linear representation of expression and its tree representation. In particular tasks propose to construct both a linear representation of expressions presented as tree and vice-versa a tree representation of a linear expression.

Tasks have to be solved first with paper and pencil and then working with Aplusix. The answer to the task have to be written in the students notebooks or in the card.

At the end of this card a discussion can be planned to explicit some concepts about the relationship between the tree representation and the linear representation of an expression that have been emerged by the solution of the tasks.Some rules about the hierarchical structure of an expression should be made explicit (priority of parentheses, multiplication sign respect to addition, etc.).

Card 3

The tasks proposed in this card focus on the observation that it is possible to find different tree representations of a linear expression while the opposite case (to construct different linear representations from a tree representation) it is always not possible.

Card 4

This card focus on different repreentation of a numerical expression. In particular tasks concern the passage from natural language to the tree or linear representation of the expression.

Tasks have to be solved first with paper and pencil and then working with Aplusix. The answer to the task have to be written in the students notebooks or in the card.

At the end of this card a discussion can be planned to explicit how a statement of an equation, expressed in natural language, can be transformed in algebraic language (by a tree representation or a linear representation).

<u>Examples from the classroom</u>

In general, student achievement is revealed by analysis of student protocols and comparison of initial and final test results. These results have revealed some development in students' procedural competencies regarding numerical expressions with respect to the initial test results. At the end of the experiment some students were able to manage the "systemic" structure of expressions: they could identify the main operation of an expression, they could understand the priority of some operations with respect to others, and could sometimes identify equivalent expressions.

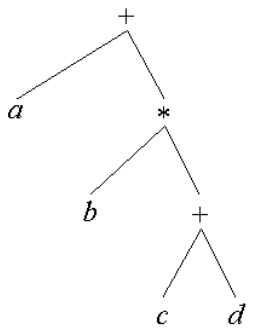
With respect to the educational aim the following results were observed in the final test analysis:

- The majority of students were able to build tree representations for numerical expressions. Note that at the beginning of the experiment students were unfamiliar with tree representations of linear expressions.
- Most students learned how to "read" an expression represented by a tree, at least for simple expressions. Nevertheless, some difficulties concerning the use of parentheses for expression translation are still evident in students' behaviour: some students are unable to control parentheses even if they are able to do calculus correctly (See test results)
- The language used by students to describe expressions is clearer and more appropriate with respect to the initial test. In the initial test, students translate the expression respecting the "stenographic" way of the expression at hand, while in the final test some students are able to use the ideographic nature of algebraic language. For example, in the final test, Giulia described the expression 2*(3-1)+4/2 by writing "two times the difference between 3 and 1 is added to the quotient between 4 and 2". In the initial test she described the expression (5-1)/4 writing "I subtract 5-1 and I divide the result by 4".
- Some students learned to identify equivalent expressions from a structural point of view by comparing tree representations and linear representation of an expression.

Possible extension

The analysis of students' solutions has highlighted that, opposite to our expectations, the second task of the Card 2 was easier respect to the first one. The difficulties emerged in the second task mainly depend on the poor experience of students in the tree construction. On the contrary, in the students' solutions of the first task we have found many mistakes that are not present in those of the second task. These mistakes depend on the use of parentheses: many students wrote the linear expression without using them, even when they were necessary. To explain this fact, a first consideration is that when student has to translate a tree representation into a linear expression he has to choose if insert parentheses or not, while when he has to construct a tree starting from a linear expression he has to translate parentheses but not insert them in the tree. A deeper analysis highlights that to accomplish the second task student has to know the syntactical structure of the tree (how to build a tree) and has to respect some computational rules. Student has to build the tree taking into account that collapsing bottom-up the tree he has to find the sequence of computation described by the linear expression. This task strengthens procedural skills, or in other words the "superficial structure" of numerical expression. Opposite, to accomplish the first task procedural skills are not sufficient. Student has to interpret the tree structure.

Consider the following tree:

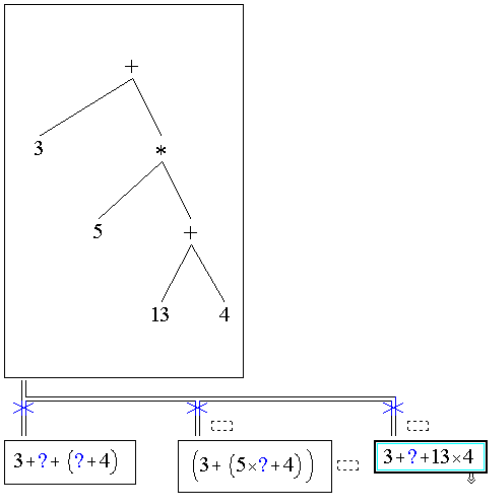


If student read the tree in procedural way, he could wrong in choosing between these three expressions: a+b*(c+d) or a+(b*(c+d)) or a+b*c+d. In order to convert the tree in linear form inserting parentheses in the correct place, it is important to read the tree interpreting its systemic structure and this entails the capability to manage the numerical expressions in structural way. For this reason, a-posteriori we think that it would have been perhaps more appropriate to propose students the second task before the first one.

Moreover, A task that has produced interesting didactical results in learning, was the task 2 in the Card 3 (in which students have to complete tree representations in given diagrams and verify their answers in Aplusix).

This task is quite unusual in the experience with Aplusix and seems to be cognitively richer than the others because it obliges students to consider the structure of the expression. We think that it is an interesting task because to solve it students have to focus the attention on structural aspects of a numeric expression. Students have to interpret the representations assigned with the task and to compare among them. Through their comparison students receive hints that orient them to focus the attention on structural aspect of the numerical expression to replace the question mark. In this solution the feedback is crucial.

Thus, in a future scenario other similar tasks might be inserted. For example, a tree could be proposed along with some incomplete numerical expressions, and students could be asked to complete these where possible so that the expression is equivalent to the tree representation. (See figure below).



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