## 1. Title: Exploring equations

## 2. Scenario identity

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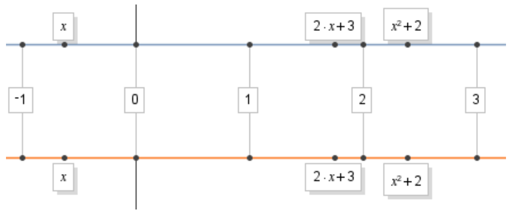



**Subject Area:** Elementary Algebra

**Topic(s)**: Algebraic equality and solution of equations

## 3. Activity rationale

### Teaching and Learning problem addressed

What does solving an equation mean? What could we intend for truth value of an equality? What does it mean that two equations are equivalent? What is an algebraic identity and what differentiates it from a conditioned equality? The aim of this scenario is to promote the conceptual development and the construction of meanings involved in answering these questions and in solving equations.

Important conceptual developments are needed to pass from the use of numerical expressions and propositions of arithmetic to literal expressions and propositions of elementary algebra. As a matter of fact, in arithmetic only numbers and symbols of operations are used and the control of what expressions and propositions they denote can be realized through some simple computations. In elementary algebra, instead, letters are used to denote numbers in an indeterminate way and new conceptualisations are necessary to maintain an operative, semantic and structural control on what expressions and propositions denote. The need for this conceptual development emerges clearly with the construction of the notion of algebraic equality. On the morphological plan, an equality is a writing composed by two expressions or by an expression and a number connected by the sign "=". On the semantic plan, equality denotes a truth value (true/false) related to the statement of the comparison. When the expression(s) composing the equality is (are) strictly numerical, it is easy to verify its truth value through some simple calculations (e.g., 2*3+2=8 is true while 2*3+2=9 is false). Experiences with numerical equality contribute to structure a sense of computational result for the "=" sign. This sense can be an obstacle in the conceptualisation of algebraic equality as relation between two terms, as highlighted by several research studies (Arzarello et al, 2001; Filloy et al., 2000). When the expression(s) composing the equality is (are) literal the equality can present different meanings because the value assumed by the letter can condition differently its truth value. In these cases the sign "=" should suggest to verify the numerical conditions of the variable for which its two terms are equal. There are cases where the two terms could never be equal whatever the value of the letter, as in this example: 2(x+3)=4x-2(x-1). In other cases to interpret an equality on the semantic plane, it is necessary to distinguish if it has to be considered as equation or as identity. The sign "=" assigns to the equality the sense of equation when its two members are equal only for specific values of the letter. For example, the equality 2x-5=x-1 is true only for x=4 and it is false for all the other values. Instead, the sign "=" gives to the equality the sense of identity when its two members are equal whatever the numerical value of the letter, as for example in this equality 2*x+1=x+(x+1). In order to master the algebraic equality, a conceptual development relating to notions such as those of equation, identity, truth value, truth set and equivalent equation is necessary. Moreover, to express the way in which the letter can condition the truth value of an equality, it is necessary to develop the ability to use universal and existential quantifiers, though in an implicit way.

### Innovation:

Errors emerging in the solution of equations always reflect difficulties of semantic and structural nature. For this reason, in this scenario the logic and formal approach in the equations solution is strictly connected to the

quantitative approach. This connection is aimed to build the semantic and structural competencies needed to justify the formal operability.
This scenario is centred on the use of Alnuset system and, in particular, of the Algebraic Line and the Symbolic Manipulator which are embodied in it. This scenario exploits the new representative and operative possibilities of these two environment of Alnuset to integrate the quantitative, logic and formal approach in the didactical practice thus promoting the conceptual development and the construction of the meanings related to the notions of algebraic equalities and of solution of an equation.

### Added value

Many representative events which emerge on the Algebraic Line of Alnuset can be related to the concepts and algebraic notions to be developed. They can be effectively used to verify the preservation of the equivalence in the logic and formal transformation of algebraic expressions and propositions.

## 4. Context of Implementation

### Educational Goals

The didactical goals of this activity concern the conceptual development and the construction of meanings in relation with the following notions:

- conditioned equality,
- solution of an equation,
- equivalent equations,
- truth value of an equality
- truth set of an equation
- algebraic identity

### Which students?

$10^{th}$ grade. The scenario may also be appropriate also for the $9^{th}$ grade.

### Students prerequisites

Basic familiarity with Alnuset environments and in particular with the operative and representative opportunities of the Algebraic line of Alnuset. More in particular, in our experimentation this pedagogical scenario on equations has been developed after twelve hours of activities on properties of the algebraic expressions and of polynomials based on the use of Alnuset

### Duration

3 school hours

### Place

The school computer lab. Students work in pairs. Each pair works on a computer with Alnuset and produces only one protocol containing the answers to the activity.

### Resource and tools

Alnuset and worksheet.

Alnuset is constituted of three strictly integrated components: Algebraic Line component, Symbolic Manipulator component, Functions component. In this activity only the first two components are used. For this reason, here we will describe only the Algebraic Line component and the Symbolic Manipulator component. They make available both techniques of quantitative and symbolic nature to operate with algebraic expressions and propositions.

The main characteristic of **Algebraic Line** component is the possibility to represent an algebraic variable as a mobile point on the line, namely, a point which can be dragged with the mouse along the line. When dragging the variable mobile point on the line, all algebraic expressions containing such a variable move accordingly. This operative and representative characteristic of the Algebraic Line allows users to explore both properties of the algebraic expressions and relations among them. Moreover, in this environment techniques of quantitative nature to find roots of polynomials with integer coefficients and to explore and validate the truth values of an algebraic proposition are available. The characteristics and the quantitative techniques of the Algebraic line make a dynamic algebra possible. We note that the above described three representative events of the algebraic line are of great importance for the didactical goals of this scenario.

- The belonging of two expressions to the same post-it can be connected to the notion of equality and equivalence between expressions. As a matter of the fact, the belonging of two expressions to the same post-it (for a value assumed by the variable on the line) is index of their equality. If two expressions are contained in the same post-it for all values assumed by the variable on the line, they are equivalent.
- The representative events which emerge with the application of the E=0 function to a polynomial represented on the line and with the drag of the variable on the line can be connected to the notion of polynomial root (see task 4).
- The colour accordance between the marker associated to a proposition and the marker associated to the numerical set constructed by the user, can be connected to the notion of truth set of the proposition and can be used to validate the constructed numerical set as the truth set of the proposition.

The main characteristic of **Symbolic Manipulator** component is the possibility to operate with algebraic expressions and propositions in formal and logic form through a structured set of manipulative commands which preserve the equivalence through the transformation of their form. (Fig. 1). These commands correspond to basic properties of operations, to the properties of equalities and inequalities, to the logic operations among propositions and to the operations among sets. Another characteristic of the Manipulator is the possibility to create a new transformation rule (user rule) once this rule has been proved. The user can easily control the whole process of algebraic transformation exploiting feedback given by the system. Moreover, the user can verify the preservation of the equivalence in the transformation representing the transformed forms on the Algebraic Line.

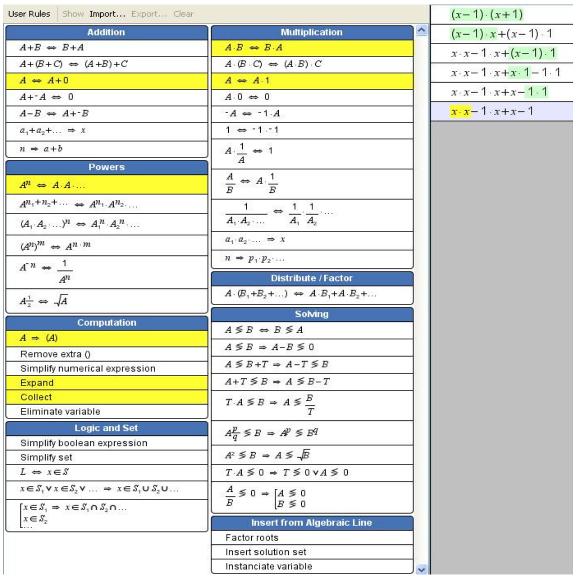



Fig. 2 Symbolic manipulation of the expression (x-1)*(x+1) by means of the rules available on the interface of the Symbolic Manipulator of Alnuset as regards to the selected part of the expression.

## Types of activities

The operative and representative possibilities of the algebraic line environment have been exploited to design explorative activities that address the construction of meanings for the notion involved in the solution of a second degree equation.

## 5. Enactement of the activity

This scenario is composed of seven activities that focus on the solution of a second-degree equation. The formula to solve second degree equations is not required. For each activity we present the task assigned to the student, we describe the mediation role of the tool used (Alnuset)

**Activity 1**

*Consider the following two polynomials: $x^2$+2; 2*x+3. Explain what does it mean putting the equal sign between them, or, in other words, explain how you interpret the following writing $x^2$+2=2*x+3. Then represent on the Algebraic Line of Alnuset the two expressions $x^2$+2 and 2*x+3 and drag the mobile point x to verify your hypothesis.*

### Resources and tools

The first part of the task is aimed to make explicit students' conceptions on the notion of algebraic equality. The research activity has highlighted that many students attribute the meaning of result of a computation to the sign"=". Coherently with this meaning, we expect that the equal sign between two expressions could suggest that the computation related to the two terms of the equality has to produce the same result whatever the values of the variable x is.

The second part of the task aims to discuss this misconception and to construct the idea that the equality between two members is conditioned by the value of x. Thus, the second part of the task focuses on the use of Alnuset. Students are requested to drag the mobile point x along the line and to observe that the points corresponding to the two expressions move accordingly. In this way students con observe that there are only two values of x for which the points of the two expressions are close to each other, almost coincident (Fig. 2, Fig. 3, Fig. 4).

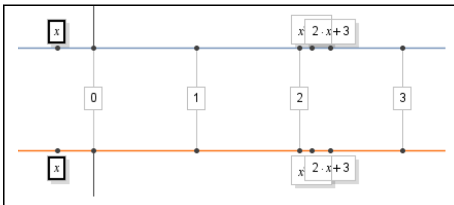



Fig. 2

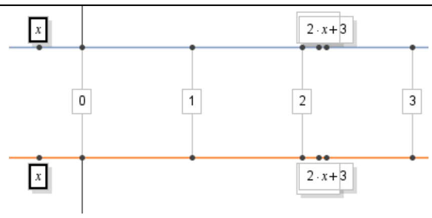



Fig. 3

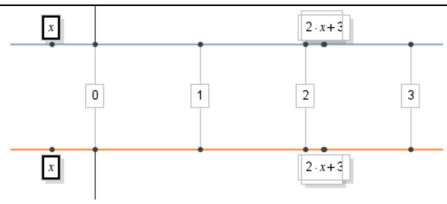



Fig. 4

Since these numerical values of the variable are irrational and have to be constructed on the line, students cannot verify directly that the two expressions are exactly coincident on the same point and that they belong to the same post-it (see Fig. 5 ).

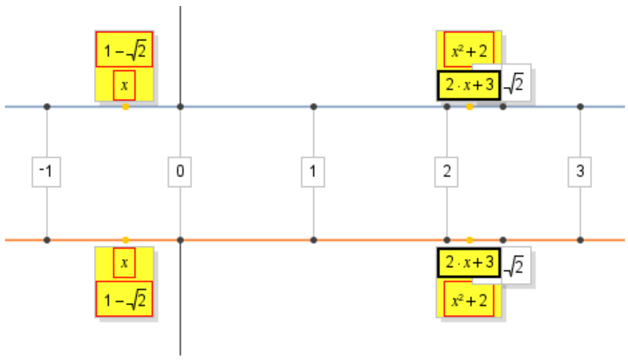


Fig. 5 For x=$1-\sqrt{2}$ the proposition $x^2+2=2*x+3$ is true. In others words, when x assumes the value $1-\sqrt{2}$ the expression $x^2+2$ and $2*x+3$ correspond to the same point and they belong to the same post-it.

The technique mediated by Alnuset to find these irrational numbers requires the transformation of the equation $x^2+2=2*x+$ in its canonical form ($x^2-2*x-1=0$), the representation of its associated polynomial on the line and the use of the E=0 command to find the roots of this polynomial. The transformation of the equation in canonical form is realized in the Symbolic Manipulator and it is guided by the following task.

## Activity 2

*Select the equation* $x^2+2=2*x+3$ *and use the rule A = B ⇔ A-B = 0 of the Symbolic Manipulator to transform it. Translate the result produced by this rule into natural language.*

### Resources and tools

This task focuses on the rule A = B ⇔ A-B = 0 of the Manipulator through which it is possible to transform the equality preserving the equivalence. Once the rule is applied on the equality $x^2+2=2*x+3$ and the result $x^2+2-(2*x+3)=0$ appears, the Symbolic Manipulator makes available two possibilities to transform the proposition $x^2+2-(2*x+3)=0$ into its canonical form:

1. Using the available rules to transform step by step the expression $x^2+2-(2*x+3)$ into the expression $x^2-2*x-1$ (Fig.6)

2. Using the command "Expand" trough which the expression $x^2+2-(2*x+3)$ is automatically transformed in the expression $x^2-2*x-1$ (Fig. 7)

The choice between these two modalities depends on the goals the teacher wants to pursue and on the students' capacity to control the result produced by the computer in the two modalities. Clearly, the first strategy needs more time to be realized.

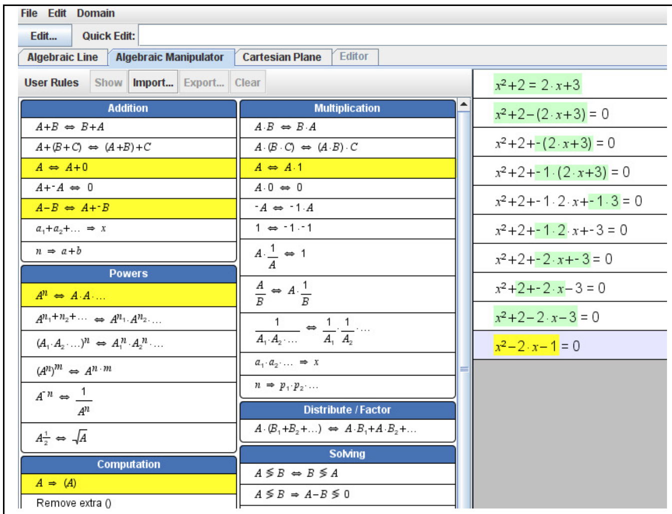



Fig.6

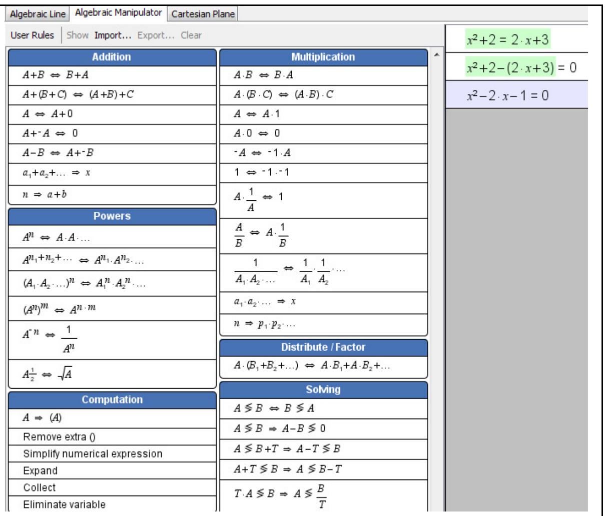



Fig.7

## Activity 3

*If you drag x along the line, what do you think could happen to the points corresponding to $x^2$-2*x-1 and 0 when the points corresponding to $x^2$+2 and 2*x+3 seem to be coincident? Make an hypothesis about the relationship among $x^2$-2*x-1 and 0 and among $x^2$+2 and 2*x+3 and try to justify it. Use the Algebraic Line to verify your hypothesis*

### Resources and tools

This task is aimed to introduce students to the notion of equivalence equations understanding that two equalities are equivalent when they are true for the same values of the variable. As a matter of fact, exploiting the drag of the variable point x along the line, students can experience that to find the values of x for which $x^2$+2 is equal to 2*x+3, it is sufficient to find the values of x for which $x^2$-2*x-1 is equal to 0. This experience is crucial to develop the notion of equivalent equations (see Fig. 8, Fig. 9, Fig 10).

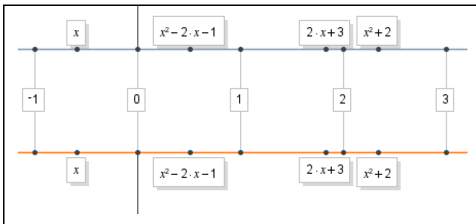



Fig. 8

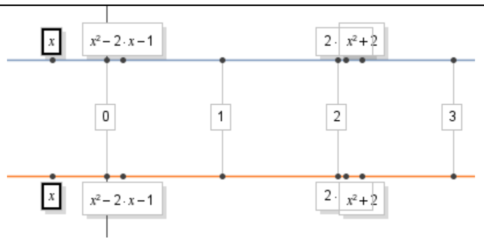



Fig. 9

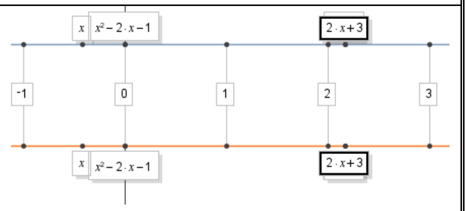



Fig. 10

Moreover, this experience is crucial to justify the necessity of the symbolic transformation of the equation. To this regard it is important to note that Alnuset made available a quantitative technique through the command E=0 which allows users to find the roots of the polynomial associated to the canonical form of the equation.

## <u>Activity 4</u>

*Though the command "send to line" available in the Symbolic Manipulator, send the equation $x^2$-2*x-1=0 and the polynomial $x^2$-2*x-1 to the Algebraic Line of Alnuset. Use the command E=0 to find the values of x which make the polynomial $x^2$-2*x-1 equal to 0.*

### Resources and tools

The command E=0 provides a quantitative and dynamic technique to find the roots of a polynomial with integer coefficients. This technique consists in applying this command to the polynomial and in dragging the variable point along the line in order to approximate the value of the polynomial to 0 (Fig. 11 and Fig. 12). When this approximation is realized, the control of the procedure pass to the system which provides a graphical feedback (Fig. 13) and it calculates the exact value of the polynomial root (Fig. 14) which is represented as a point on the line.

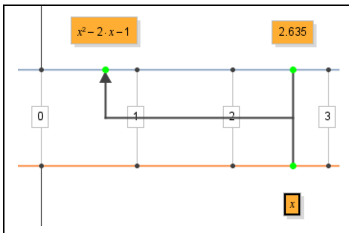



Fig. 11

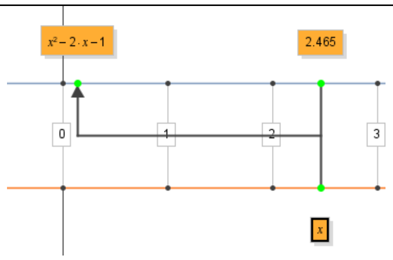



Fig. 12

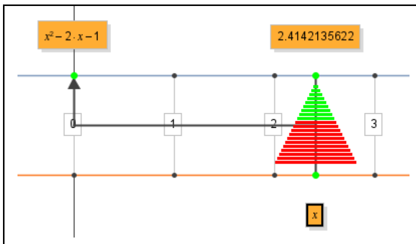



Fig. 13

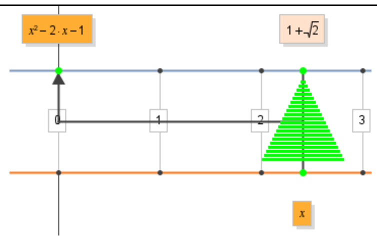



Fig. 14

Once the irrational roots are found, students can verify that, for those values of x, the point corresponding to $x^2$-2*x-1 is coincident with 0 . Moreover, students can verify that the point corresponding to $x^2$+2 and 2*x+3 are coincident and that these two expressions belong to the same post-it.

## <u>Activity 5</u>

*Use the function "Edit Set"of the Algbraic Line to define the set of values which make the equality $x^2$-2*x-1=0 true. Drag the variable x along the line to validate the truth set constructed.*

### Resources and tools

In the algebraic line the expressions are represented on the line and they are associated to points on the line while equalities are represented in a specific window named "sets" and they are associated to a marker whose colour (red/green) is managed automatically by the system. The marker is green (Fig. 16) if, for the actual value of the variable on the line, the equality is true and, conversely, it is red (Fig. 15) if, for that value of the variable, the equality is false. The drag of the variable point allowed students to explore the truth of the equalities and to construct a meaning for this notion.

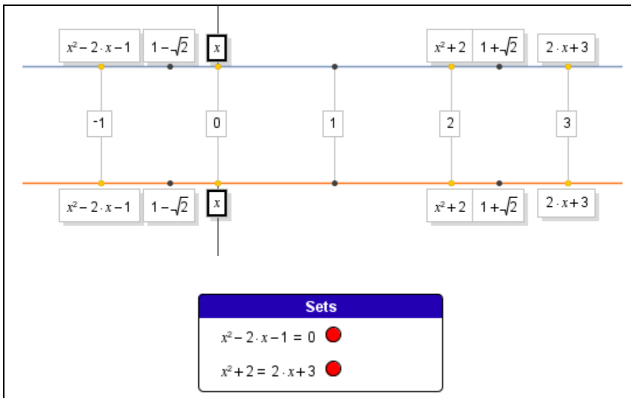



Fig. 15

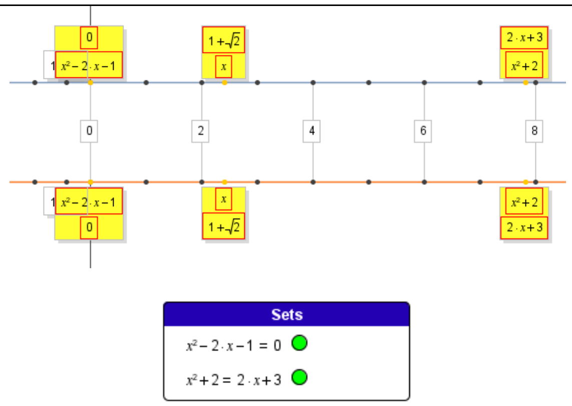
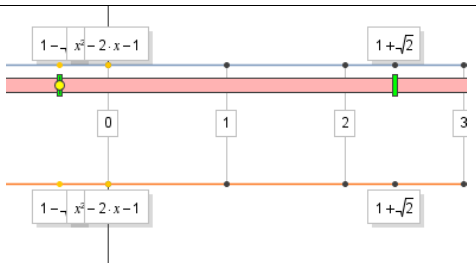
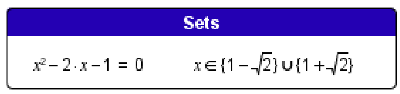



Fig. 16

To experience and build the notion of truth set of an equation, two other functions available in the Algebraic Line can be exploited: a graphic editor, through which it is possible to build the truth set of an equality, and a particular feedback of the system to validate it.

Once the polynomial roots have been represented on the line, the user can use the graphic editor to construct the truth set of that equation (Fig. 17). Such graphic editor allows the user to operate on the line in order to define a numerical set that the system automatically translates into the formal set language (Fig 18).



| Fig. 17 | Fig. 18 |
|---|---|

This truth set of the proposition can be validated using the coloured dots. As a matter of fact, by dragging the variable along the line, the colour accordance between the dot of the proposition and the dot of the built set, allow the user to validate the defined numerical set as truth set of the proposition (see Fig. 19 and Fig. 20). Moreover, the validation process is supported by the feedback of quantitative nature (see post-it) provided by the system with respect to the position of the variable point and of the polynomial point on the line during the drag of x.

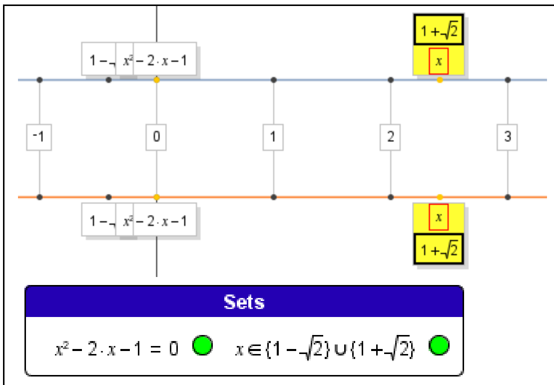



Fig. 19

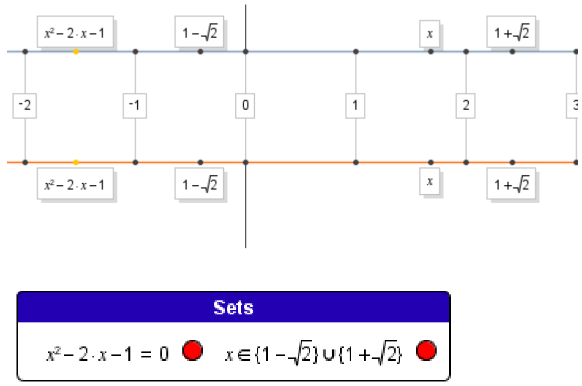


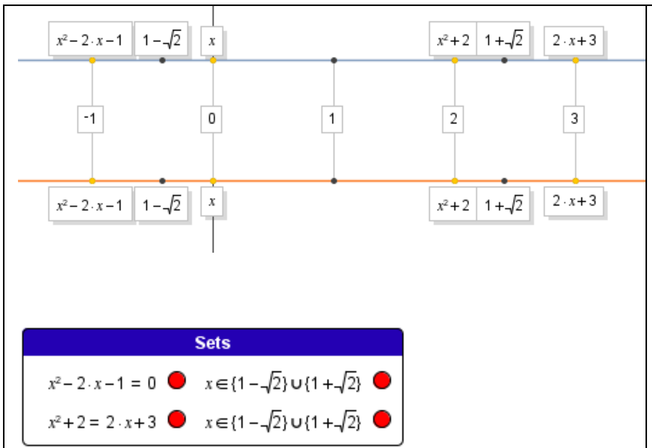

Fig. 20

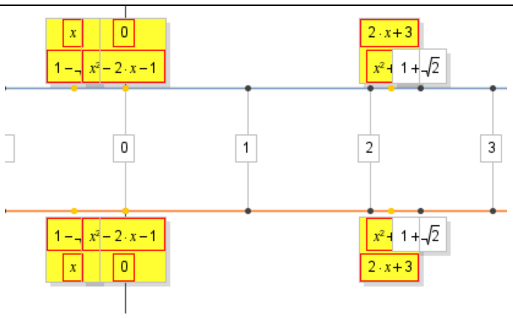
## Activity 6

*Explain what is the relation between the solution of the equation $x^2$-2*x-1=0 and of the equation $x^2$+2=2*x+3. Without using the Algebraic Line, write what you think about this statement: " For the found values of x, the expressions $x^2$-2*x-1, $x^2$+2and 2*x+3 are coincident in the same point and this point is 0".*
*Use Algebraic Line to verify and justify your answers.*

### Resources and tools

This task aims to orient towards the institutionalisation of the notions of conditioned equality and of equivalent equation. The presented statement reflects a common misconception concerning the equivalent equations: two equations are equivalent when all their members are equal for particular values of the variable. The use of Alnuset is aimed to destabilize this misconception focusing the attention on the fact that equivalent equations are characterized by the same truth set. Dragging the variable point along the line (Fig. 21), students can dynamically verify that only for the values $1+\sqrt{2}$ and $1-\sqrt{2}$ (Fig22) of x there is accordance of the green colour between the markers associated to the equations and those associated to their respective truth sets. Moreover they can verify that for those same value of x the point corresponding to $x^2$+2 coincides with the point corresponding to 2*x+3 and $x^2$-2*x-1 coincides with 0.



| Fig. 18 | Fig. 19 |
|---|---|
| | **Sets**<br>$x^2-2\cdot x-1=0$ ● $x\in\{1-\sqrt{2}\}\cup\{1+\sqrt{2}\}$ ●<br>$x^2+2=2\cdot x+3$ ● $x\in\{1-\sqrt{2}\}\cup\{1+\sqrt{2}\}$ ● |

Since the main aim of this activity is to develop an operative, structural and semantic control on the algebraic equality, it is necessary that students can distinguish the conditioned equality from the algebraic identity. Up to now tasks have been designed in order to construct the notion of equation as equality that is conditioned by values of the variable.

The following task is aimed to construct the notion of algebraic identity

## <u>Activity 7</u>

*What can you say about the truth set of an equality composed by two equivalent expressions? Write your hypothesis.*
*Consider the following equality:*
*$x^2+1=(x-1)^2+2*x$*
*Use Algebraic Line and Symbolic Manipulator to verify your hypothesis.*

### Resources and tools

The aim of the first part of the task is to make explicit the students' hypothesis concerning the equality between two equivalent expressions. We expect that, to answer this question, students exploit all the algebraic notions previously dealt with in this educational activity (truth value of an equation, truth set of an equation, equivalent equation). In the second part of the task we ask students to validate their hypothesis using Alnuset and referring to a particular algebraic equality. The task asks to verify the hypothesis both on the quantitative plane, using the Algebraic Line component, and on the formal and logic plane, using the Symbolic Manipulator component.

On the Algebraic Line component students can experience that dragging the variable point along the line:

- the points corresponding to the expressions $x^2+1$ and $(x-1)^2+2*x$ are coincident and the two expressions belong to the same post-it.
- The colour of the marker associated to the equality $x^2+1=(x-1)^2+2*x$ is always green
- The accordance of the green colour between the marker corresponding to the equality $x^2+1=(x-1)^2+2*x$ and that corresponding to the built truth set is maintained (Fig. 23)

| **Sets** | |
|---|---|
| $x^2+1=(x-1)^2+2\cdot x$ ● | $x\in\,]-\infty,+\infty[$ ● |

Fig 23 Accordance of the green colour between the martker corresponding to the equality $x^2+1=(x-1)^2+2*x$ and that corresponding to the built truth set for every value of x

Moreover, in Symbolic Manipulator, students can transform the equation demonstrating the truth of the equality

$$x^2+1=(x-1)^2+2\cdot x$$
$$x^2+1=x^2-2\cdot x+1+2\cdot x$$
$$x^2+1=x^2+1$$
$$x^2+1-(x^2+1)=0$$
$$0=0$$
*True*

Fig. 24.

Moreover , through the command “Insert solution set” (Fig. 25) that can be applied to the equation only if the user has previously constructed the truth set of the equality on the Algebraic Line component, it is possible to express the solution in terms of this truth set.

$$x^2+1 = (x-1)^2+2\cdot x$$

$$x \in ]-\infty,+\infty[$$

Fig. 25

**Organisation of classroom activity**

This sequence of activities has been designed to engage students to work collaboratively in groups of two. We think that on the basis of this organisation it is possible to orchestrate prosperous circumstances for discussing, forming and testing hypotheses, developing strategies and negotiating. Within the activities students can also participate in discussions guided by the teacher to share their ideas with other workgroup members, to compare their solution strategies or their interpretations of the representative events emerging in the use of Alnuset .

**Assessment suggestion**

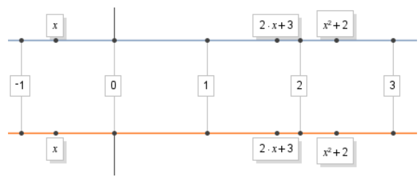
At the end of the sequence of activities we suggest to assign other second degree equations to students with the request to find their solutions explaining the performed solution steps in the Algebraic line and Manipulator environments and justifying them at operative and semantic level

## 6. Examples from the classroom

To give an idea of the mediation role of the Algebraic line in the construction of algebraic meaning , let us consider as an example some results taken from of our experimentation related to the task 1 and task 5

In the their answers to the question of this task, many students attribute to the “=” sign the meaning of computation result. Nevertheless they were already faced with 1st degree equations. A typical students’ answer is: “*To put the equal sign between two polynomial expressions means that these expressions have the same result*”. For many students inserting the equal sign between two expressions suggests the idea that the computation result of the two terms has to be equal when a value is assigned to the letter.

In the following task students were asked to represent the two expressions on the algebraic line of Alnuset to verify their answers. Dragging the mobile point x along the line (and observing that the points corresponding to the two expressions move accordingly), all students noted that there are only two values of x for which the points of the two expressions are close to each other, almost coincident. Through this exploration students experienced that equality of two expressions is conditioned by numerical values of the variable, which is crucial to develop the conditioned equality notion. In previous activities with Alnuset, students experienced that every point of the algebraic line is associated to a post-it that contains all expressions constructed by the user denoting that point. In order to verify equality of two expressions, the students tried to find values of x for which the two expressions belong to the same post-it. Since these irrational values had to be constructed on the line, the students could not verify this directly: “*we don’t understand what is the number…it will be 2 point something…even if we use zoom in we don’t understand …*”. The construction of the solution occurred through the successive tasks



## 7. Possible extension

At curricular level many extensions are possible. Here we suggest only two proposals that can be of great interest for the education perspective:

* Using the Manipulator environment of Alnuset to demonstrate the formula to solve second degree equations with the aim to create a new user rule of solution that can be successively used in other activities

* Using the Algebraic line and the Manipulator environments to solve second degree equations with parameter and to perform a discussion of the solution on the basis of the value assumed by the parameter

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