Annexes

A.1 Analysis of the Teaching Experiment

A.2 On the actual implementation of the TEs

A.2.1 Questionnaire: On the actual implementation of the TEs

Experimenting Team:

Teaching Experiment Title:

DDA:

1. (**Quantitative infos**)

Number of classes involved:

For each class, specify:

a. Kind of school ________________, grade_____________, age______________

b. Number of hours: ___________________

c. (Approx.) Date of beginning: __________________________________

d. (Approx.) Date of ending: ______________________________

C*omments:*

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

- ☐ No, no variations at all
- ☐ Just small adjustments
- ☐ Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

A.2.2 Didirem TE with Casyopée

1. (**Quantitative infos**)

Number of classes involved: 2

For each class, specify: (same structure in the two classes)

a. Kind of school ___Lycée_______, grade_11____, age_17____

b. Number of hours: _______10__________

c. (Approx.) Date of beginning: ____Oct. 2007____________________________

d. (Approx.) Date of ending: _______Dec. 2007______________________

Comments:

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

X No, no variations at all

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

No

A.2.3 Didirem TE with Casyopée

1. (**Quantitative infos**)

Number of classes involved: 2

For each class, specify: (same structure in the two classes)

a. Kind of school ___Lycée_______, grade_11____, age_17____

b. Number of hours: _______5___________

c. (Approx.) Date of beginning: ____Sept. 2007____________________________

d. (Approx.) Date of ending: _______Oct. 2007______________________

Comments:

Another experiment took place with younger students in spring 2008 and with the same Educational Goals. The tasks were similar, but the activities were carried out outside the classroom.

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

X Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

The teaching sequence was designed in two phases: one of familiarization with Cruislet where we would try to show its potential for raising and working interesting issues in the "travaux personnels encadrés" (TPE, personal project work with the support of teachers) perspective, and one of use of the DDA within specific projects whose theme had to be fixed by the students themselves according to the TPE didactic contract. This pedagogical plan had to be amended on two important aspects, explained below.

1. We modified the work in the first phase when preparing the sessions with the teachers

The two teachers have been involved in the Casyopée project and working in collaboration with the Didirem research team for 10 years. They had a good understanding of the ReMath objectives and were ready to invest time and energy for experimenting Cruislet, while they also thought that the institutional distance with the mathematics French curriculum was big.

They participated in the elaboration of the pedagogical plan. They expressed concerns about the difficulty of bringing activities not directly consistent with the curriculum, in the mathematics course (scientific stream) at 11^{th} grade where the syllabus to cover is very heavy. They agreed with us that using the TPE structure would certainly be the right choice while insisting on the fact that the "TPE spirit" of free choice by the students and of consistency with the national themes should be preserved, limiting the space of freedom for the experiment.

During the year 2006-2007 they tried to acquire knowledge about Cruislet, but difficulties were experienced with the first version. Researchers and teachers had difficulties to get aware of the potentialities and we rather shared their interrogations. Thus we could not go very far into the teaching sequence design until we received the second version and the correction to the graphic display (may 2007). Also the DDA manual listed only a small subset of the LOGO commands and no control instructions (Make, repeat…).

After that, we prepared the three sessions devoted to classroom work directed by the teacher for the first phase. When the teachers worked on this scenario with the version of the DDA they had received, it quickly appeared that the tasks proposed in the PPM for the first phase were not enough specified and were also too ambitious. We had thus to revise the scenario and design tasks that, based on the same ideas, could be understood by the students as consistent with the curriculum, as well as feasible.

We conceived thus two central tasks that correspond to the first steps of the PP (presentation of the software and preparing and programming trips):

1. Prepare a flight from one town to another staying as close as possible from the ground.

This implies computing angles in vertical planes that allow the avatar reach a point without crashing on the ground.

2. Prepare Logo procedures to make an avatar flight in a given figure (circles, spirals, helixes..)

We thought of combining the two tasks (for instance circle around the Mount Olymp) but could not implement this idea because of time constraints.

The task of correcting a trajectory to cope with the wind effect was appealing and really could have make us take advantage of the Cruislet's potential for treating vectors. We nevertheless abandoned this last part of the scenario for two reasons:

1. Teachers thought that the two tasks above were already enough time consuming and that they would certainly be sufficient for ensuring the required familiarity with Cruislet for using it in projects.
2. We could not find a way to have a feedback, allowing students to know whether they accurately corrected the trajectory. It would have been necessary to build into Cruislet some feature for this kind of effect.

2 The second phase could not be implemented

The sessions in the first phase ran reasonably well and were quite interesting. A geography teacher participated in the presentation of Cruislet bringing information about the geography of Greece and the systems of representation.

Then the teachers invited students to design their own project, in the frame of the TPE, and tried to encourage projects around Cruislet and the modeling of 3D deplacements. They did not succeed. The problem was that students were attracted by other themes than modelisation for their project of TPE. Actually, this theme is viewed as difficult by students and rarely chosen. The attractive power of Cruislet was not strong enough for overcoming this obstacle.

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

Because of the above reasons, the experiment went differently from what was planned in the guidelines. Thus it is necessary to adapt the goals, hypothesis and research question.

Educational Goals

Use Cruislet's potential for students working on 3D geographical representations and displacements. Reinvest trigonometric notions in non-standard settings (calculations of angles for displacements).

Introduce students into Logo programming, giving sense to Logo procedures as a record of deplacements and proposing attractive problems, while not too far from the curriculum, as well as feasible

Educational Hypothesis

It is possible to conceive tasks to introduce students to this complex software and that appear to students consistent with the curriculum and attractive. An introduction to Logo structured programming could be done by presenting students an iterative procedure as a solution to a problem they solved non-iteratively (flying in a equilateral triangle) and asking them to adapt this procedure to generate flights following other figures (circles, spirals, helixes…).

Common Research Questions

How do students appropriate and coordinate representations of displacements in Cruislet (Cartesian and Polar). Is it possible to introduce them to Logo programming in a short time? What are the consequences on the understanding of the underlying mathematical objects?

Note: as indicated above we carried out another experiment later on, in order to get more data for analysis.

A.2.4 ETL TE with Cruislet

1. (**Quantitative infos**)

Number of classes involved: 2

For each class, specify:

a. Kind of school Upper high school , grade 1st , age 15-16

b. Number of hours: 28 (1st class), 8 (2nd class)

c. (Approx.) Date of beginning: 19/10/2007 (both classes)

d. (Approx.) Date of ending: 11/4/2008 (1st class), 14/12/07 (2nd class)

Comments:

- Due to context problems there was an extension of the period of TE implementation.

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

□ No, no variations at all

√ Just small adjustments

□ Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

Teaching Sequence (1st class):

- Kind of tasks: The familiarisation phase was extended to 8 hours and several tasks were added.
- Additional activity sheets were created.
- The time schedule was extended.

Teaching Sequence (2nd class):

- Kind of tasks: The teacher changed some of the tasks, as she wanted to focus only at the concept of function.
- Additional activity sheets were created by the teacher.
- Time schedule: The second TS lasted 8 hours.

The changes made, mainly affected the PP structure, as the teacher of the 2nd TE altered the PP to fit her own educational goals (focus mainly at the concept of function rather on navigational mathematics).

The hours of implementation of the 1st TE were much more than the 2nd one. As a result, this had a major effect in reformulating and answering the SRQs as most data were not primarily focused at the concept of function, but at navigational mathematics as well (e.g. emerging mathematical concepts, coordinates). Thus, in our analysis we preferred to combine the results of both TEs in the CRQ (regarding the concept of function) and reformulate the other SRQs.

A.2.5 ETL TE with MaLT

1. (**Quantitative infos**)

Number of classes involved:

For each class, specify:

a. Kind of school ___***Public multi-cultural high school***________, grade___***7th***_________, age___***13***___________

b. Number of hours: ________***18***___________

c. (Approx.) Date of beginning: ______***01/11/2007***____________________________

d. (Approx.) Date of ending: _______***19/12/2007***_______________

Comments:

Each teaching session (hour) lasted 45-50 minutes.

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

- ☐ No, no variations at all
- ☑ Just small adjustments
- ☐ Major variations

There were no major variations in the implementation of the plan. However, small adjustments were performed which primarily concerned revision of the time schedule and modifying/omitting specific activities. Two examples:

(a) the Introductory Activity related to student's familiarisation with the environment took more time than initially planned
(b) the implementation of the activity Spiral Staircase Simulation was omitted due to time restricts of the school.

Those adjustments did not affect the possibility of answering the questions a-priori formulated in the TE portrait that we had provided.

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

A.2.6 ETL TE with MoPix

1. (**Quantitative infos**)

Number of classes involved: **1**

For each class, specify:

a. Kind of school: **Upper Secondary – Vocational Education School**, grade: **12th**, age: **17 to 21**

b. Number of hours: **25**

c. (Approx.) Date of beginning: **07/11/07**

d. (Approx.) Date of ending: **20/12/07**

Comments:

Ten sessions took place. The duration of each one was 2 to 3 hours.

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

- ☐ No, no variations at all
- ☑ Just small adjustments
- ☐ Major variations

Comments:

Although there were no major variations in the implementation of the plan, several adjustments were performed as far as the time schedule and the activities initially designed are concerned. The time-schedule was revised at least twice during the experimentation process while a number of activities described in the initial version of Pedagogical Plan were modified, moved to another phase of the experimentation or even omitted.

The possibility of answering the questions a-priori formulated in the TE portrait was not affected by any of those adjustments.

In case of major variations, report on them: which aspects have been mainly modified? What are the reasons for those variations?

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

A.2.7 IoE TE with MoPix

1. (**Quantitative infos**)

Number of classes involved: 1

For each class, specify: 7 students (Advanced Level Mathematics)

a. Kind of school ____college______, grade_____________, age__16-19_______

b. Number of hours: ________15__________

c. (Approx.) Date of beginning: ____31/10/2007 (October 2007)______________

d. (Approx.) Date of ending: _______19/12/2007 (December 2007)____________

Comments:

– Weekly meeting for 1.30 (one-and-a-half hour)

- voluntary participation: inconsistent attendance due to other student priorities reduced scope of teaching sequence
- teaching conducted by researchers

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

□ No, no variations at all

✓ Just small adjustments: More time than anticipated was spent on some aspects of the teaching sequence. As a consequence most students did not complete tasks related explicitly to graphing or interactions between objects.

□ Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

The omission of work on graphs prevented us from addressing one of the sub-questions to our reformulation of the CRQ which referred specifically to graphical representation.

A.2.8 IoE TE with MaLT

1. (**Quantitative infos**)

Number of classes involved: 1

For each class, specify: 24

a. Kind of school state, grade year 8, age 12-13

b. Number of hours: 8

c. (Approx.) Date of beginning: 20/11/2007

d. (Approx.) Date of ending: 30/11/2007

Comments:

- 8 meetings: 4 times a week with two meetings in one day

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

- □ No, no variations at all
- □ ✓ Just small adjustments The details of materials provided to support use of MaLT were developed during the teaching experiment in response to our evaluation of student needs.
- □ Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

No

A.2.9 ITD TE with Alnuset

1. (**Quantitative infos**)

Number of classes involved: 1

For each class, specify:

a. Kind of school classics Liceo, grade 10, age 15

b. Number of hours: 20

c. (Approx.) Date of beginning: 10th October 2007

d. (Approx.) Date of ending: 19th December 2007

Comments:

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

- □ No, no variations at all
- X Just small adjustments
- □ Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

A.2.10 ITD TE with Aplusix

1. (**Quantitative infos**)

Number of classes involved: 1

For each class, specify:

a. Kind of school ____low secondary school______, grade_______7______, age__12-13

b. Number of hours: __8 (+2 for pre-test and post-test)_________________

c. (Approx.) Date of beginning: ____15th November 2007____________________

d. (Approx.) Date of ending: ______6th December 2007_____________________

Comments:

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

X No, no variations at all

☐ Just small adjustments

☐ Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

A.2.11 MeTAH TE with Aplusix

1. (**Quantitative infos**)

Number of classes involved: 3

For each class, specify:

C1:

a. Kind of school: Junior High School, grade 9, age 14-15 years

b. Number of hours: four 50-minute sessions

c. (Approx.) Date of beginning: November 21, 2007

d. (Approx.) Date of ending: December 4, 2007

C2:

a. Kind of school: High school, grade 10 , age 15-16 years

b. Number of hours: 3 hours with one group (G1), 2 ¼ hours with the other group (G2)

c. (Approx.) Date of beginning: September 6, 2007

d. (Approx.) Date of ending: October 22, 2007

C3:

a. Kind of school: International High School, grade 10, age 15-16 years

b. Number of hours: four 50-minute sessions

c. (Approx.) Date of beginning: December 3, 2007

d. (Approx.) Date of ending: December 21, 2007

Comments:

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

C1, C2: Major variations

C3: Just small adjustments

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

Shortening the scenario:

- Conversion tasks RNL → RT et RU → RT were worked out with Aplusix in controlled mode only (initially, we planned to propose the same kind of activities in free mode as well) ;
- Conversion tasks RT → RNL were assigned as a homework (they were planned as classroom activities) ;
- Treatment tasks in RT were not compulsory. Teachers were free not to propose them, or propose them only to students with difficulties (Grade 10 teacher can

benefit from one hour a week that can be dedicated to remedial activities with a small number of students having difficulties in math).

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

We do not think that these modifications could affect the possibility to answer the research questions formulated a priori.

A.2.12 MeTAH TE with Alnuset

1. (**Quantitative infos**)

Number of classes involved: 1

For each class, specify:

a. Kind of school : private high school, grade 10 , age 15-16 years

b. Number of hours: 3

c. (Approx.) Date of beginning: April 10th 2008

d. (Approx.) Date of ending: April 12th 2008

Comments:

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

- Just small adjustments

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

A.2.13 Unisi TE with Alnuset

1. (**Quantitative infos**)

Number of classes involved: 2

For each class, specify:

Class 1

a. Kind of school classics Liceo, grade 9, age 14

b. Number of hours: 18

c. (Approx.) Date of beginning: 10th October 2007

d. (Approx.) Date of ending: 15th January 2008

Class 2

a. Kind of school professional school, grade 9, age 14

b. Number of hours: 20

c. (Approx.) Date of beginning: 17th October 2007

d. (Approx.) Date of ending: 15th January 2008

Comments:

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

- □ No, no variations at all
- □ Just small adjustments X (for both teaching experiments)
- □ Major variations

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

No, they did not affect such possibility.

A.2.14 Unisi TE with Casyopée

1. (**Quantitative infos**)

Number of classes involved: 4 (A, B, C, D)

For each class, specify:

a. Kind of school Scientific High School, Scientific High School, Technical School, Scientific High School

grade A, B,C: 12; D:11______, age A, B, C:17-18; D:16-17

b. Number of hours: A:12; B: 13; C: 8; D: 11

c. (Approx.) Date of beginning: A, B, C: Oct 07; D: Mar 08__________________

d. (Approx.) Date of ending: A, B, C: Dec 07; D: May 08______________

C*omments:*

2. (**Variations with respect to the designed *Teaching Sequence***) Are there any variations with respect to the *Teaching Sequence* designed before the beginning of the experimentations?

(If needed or suitable, distinguish among the different implementations)

□ No, no variations at all
[X] Just small adjustments (A,C,D)
[X] Major variations (B)

In case of major variations, report on them: which aspects have been mainly modified? what are the reasons for those variations?...

One of the experimenting teachers (B) wished to use Casyopée but was concerned about the planned number of hours (11-12 school hours). We agreed to shorten the PP (8 school hours), omitting the last two sessions.

Did those variations affect the possibility to answer the questions a-priori formulated and contained in the TE Portrait (we are referring to all the sections of the Portrait: validation of DDA and PP, CRQ, SRQ)? In case, explain how.

This decision did not affect the possibility to address the research issues stated a priori because it was possible to implement the omitted part in 3 other classes. Hence we gathered anyway enough data to answer the RQs in focus.

A.3 Questionnaire on context

A.3.1 Questionnaire answers: Didirem TE with Casyopée

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

No real problem. The DDA, the Ed. Goals and the PP were consistent with the curriculum.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

The local authorities welcomed the experiment so no problem.

Because of our choice to work with teachers who participated in the development of Casyopée, the schools were far from our laboratory, and then we had to establish a rigorous planning of the observations and to limit these.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

No problem.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain in particular issues which were unexpected.

No problem. The teaching intervention and associated observation went well.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

Both teachers were members of the research team.

A.3.2 Questionnaire answers: Didirem TE with Cruislet

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

We had concerns about the difficulty of bringing activities not directly consistent with the curriculum, in the mathematics course (scientific stream) at 11^{th} grade where the syllabus to cover is very heavy.

Then it appeared problems with the software itself. Some were corrected by interaction with Talent.

Others remained.

The teachers had also big difficulties for installing Cruislet on the schools' network. To work with students, they had to install Cruislet on a network distinguishing "administrator" and “students” rights. Then Cruislet did not open for a student session. We tried various solutions, including updating Java machines, giving rights on special directories, without effect. Teachers complained on a waste of time on technical problems remembering them their first uses of computers in schools. It appeared finally that the installer provided by Talent was not compatible with the French Windows XP because of the localisation of the Java registry keywords.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

The material difficulties we experienced in preparing the intervention have to be exposed because they show how it is sometimes not easy to pass from ‘laboratory design' to real classroom use, and the attention that has to be paid to the teachers' work conditions and workload if we aim at a successful use of technology.

Because Cruislet is based on an ambitious geographical system, an easy use requires more material resources than is the case with usual educational software. In French Lycées, the equipment is bought by regional authorities that in the present case (Brittany) privilege the quantity of computers, not the size of the memory and the up to date quality of display. This choice is generally approved especially by mathematics teachers. It generally means that students can have at home, for instance for games, computers more powerful than those they use at school.

While, after the correction of the graphic display, the computers had the minimum requirements for Cruislet use, according to the information provided by Talent, it appeared that the functioning was too slow, bringing the risk that students get bored and reject Cruislet.

The tests took a lot of time and teachers told us that they had no time to upgrade the computers. Our team had then to send a technician in the schools.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

The first sessions ran reasonably well and were quite interesting. As a first insight, students thought of Cruislet as an attractive tool, and did not complain about activities too far from the curriculum. Actually it seems that activities were found very difficult by some students, while other achieved them relatively easily. Especially concerned are: navigating with Cruislet and Logo programming.

Then the teachers invited students to design their own project, in the frame of the TPE, and tried to encourage projects around Cruislet and the modeling of 3D deplacements. They did not succeed. The problem was that students were attracted by other themes than modelisation for their project of TPE. Actually, this theme is viewed as difficult by students and rarely chosen. The attractive power of Cruislet was not strong enough for overcoming this obstacle.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

No particular problem linked to the local context.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

Both teachers were members of the research team

A.3.3 Questionnaire answers: ETL TE with Cruislet

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

- An official authorisation is needed to experiment in a school.
- Computer use for doing mathematics is not officially part of the curriculum of high school. Thus math teachers are not always willing to use computers in their class.

We found a teacher that was willing to participate to the experiment and the head of the school gave us the permission to experiment at 2 classes. Additionally the researcher attended the mathematics class for three weeks before the implementation of the TE, in order to meet the students and find favour with the teachers and the head of the school.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

- Computer lab

- Technical issues concerning the compatibility of the PCs, as the lab was not a new one.
- Difficulties to store and retrieve data to / from the computers as we were using the hypercam software in order to collect data.
- Equipment like microphones, etc. were tested.

The researcher cooperatively with the math teacher and the teacher of informatics (that was responsible for the lab) updated the machines and fixed the technical problems. This lasted for almost a month.

- Difficulties in fixing the time scedule, as we wanted too many hours for th 1st TE and the computer lab as well as the math teacher were not available.

In order to overcome this, we used an additional ‘after school’ hour during the week.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

- The head of the school changed during the school year and this affected the 1st TE in its time schedule.
- The school closed during the experimentation for almost a month, due to a sit-in by students. Thus, there was an extension of the period of implementation of the 1st TE.
- We didn’t use a camera inside the classroom although our initial aim was to use one. This was a restriction from both the teacher and the head of the school.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

The fact that the teacher was a part of our team helped us in:

- Finding a school for our experimentation.
- Extend the period of experimentation.
- Reformulating the educational activities /PP where needed.

Supporting us in coping with technical issues at the computer lab.

A.3.4 Questionnaire answers: ETL TE with MaLT

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

1. Entry to schools. The entry to schools for classroom research in Greece is very difficult. Most of the researches that are allowed by the Pedagogical Institute (the legal organisation that has the responsibility for that) are based on the use of questionnaires. The implementation of the MaLT pedagogical plan took place in the computer laboratory of a multi-cultural secondary school in Athens with one 7^{th} grade classroom (13 years olds). The access was gained through our personal contacts with teachers participating in the postgraduate course of Mathematics Education at the University of Athens. This was not new for us. Our team has long experience in conducting teaching experiments which can be considered in some way as interventions in normal educational life causing some kind of perturbation. The is perturbation would not concern only the actual educational process in the classroom involving practical issues (e.g. everyday schedules and technology use management) but would also involves much deeper issues at the socio-systemic level, e.g. teacher-student roles, social orchestration in the classroom, epistemologies and beliefs about mathematics and the educational process. All this issues were taken into account in the planning of our teaching intervention considering that they would also be part of the analysis at the institutional/cultural context level.

2. School program and curriculum. At the lower secondary level the teaching of mathematics takes place for four teaching sessions (45 minutes) a week. The school time schedule, the content and the curricular goals are determined by the National Curriculum. Although the national Curriculum suggests the use of computers concerning geometry very few teachers follow these suggestions in their teaching practice for three main reasons:
- computer use for teaching mathematics is not officially part of the curriculum;
- schools computer laboratories are usually occupied for the teaching of informatics and;
- teacher training for the use of computers in the teaching of mathematics is rather limited.

This is the main reason for which the use of computers in mathematics is not concerned with the normal school practice and thus it can be conceived as an innovation. However, the system officially gives space for teachers to enrich their lessons with activities in the computer laboratory.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

Taking into account the specificities of the global context, ETL team was deeply concerned about the specific school that the teaching experiment would take place and the teachers that would be involved. In particular it was important that the math teacher of the classroom involved would appreciate not only the added pedagogical value of the pedagogical plan's implementation but would also see it as a chance for professional development and empowerment. These concerns steered our contact with the schools and our final choice. Additionally, we negotiated with the participating teacher to embed part of the activities in the prescribed curriculum (e.g. a great part of the MaLT PPM involved the construction/properties of 2d geometrical figures which is at the core of the 7th grade geometry curriculum).

> (**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

Official delay for entry to the school. Our experimentation was delayed by the fact that we needed an official leave of entry to the school in order to carry out our teaching experiment. The time-consuming bureaucratic procedures needed has caused us problems that we hadn't anticipated in our initial planning. However, we managed to partly bypass them as a result of the excellent communication and collaboration with the school board that had let as start our experimentation before the official eave was issued.

> (**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

Two elements of the local context seemed to have raised the greatest problems in the actual implementation of our teaching intervention:

- *Students' fluency in greek.* Many students of the multi-cultural school in which our experiment took place were not really fluent in communicating in greek.

- *Students limited experience with programming.* Most of the students had not previous experience in programming with any language.

As a result the initial stages of our experimentation took more time than planned. However due to curriculum constrains the school was unable to offer more time for the experiment. This resulted in modifying/omitting the implementation of specific -initially planned-activities.

> (**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

In the MALT experiment both the experimenting teacher and the classroom teacher were members of the ETL research team. The experimenting teacher was a postgraduate student of Mathematics Education at the University of Athens while the classroom teacher had an MA degree in Mathematics Education from the same university. During the implementation of the pedagogical plan in the classroom the experimenting teacher acted also as a researcher (participant observer) and in parallel with the teaching responsibility of the classroom was also active in data collection. During the lesson the classroom teacher –also member of the ETL team - was present with a more supportive role in data collection acting as co-researcher. The fact that both teachers were members of the ETL team had positive effects on the planning of the pedagogical as well as on its implementation and data collection.

The experimenting teacher's participation in MaLT pedagogical plan construction from the design phase resulted in the development of a common communicational ground between the members of the ETL team which facilitated the implementation of the activities in the classroom and fostered the experimenting teacher's engagement in having the role of the teacher who acts as a researcher. Moreover the classroom teacher had an epistemological stance –partly due to its Ma studies- that it was compatible to the rationale and the theoretical framework underpinning our research. Thus, the classroom teacher was willing to change her classroom role to that of a facilitator, trying to elicit students' ideas and to promote dialogue and collaboration, which had affected the quality of the collected data. Weekly meetings of all members of the ETL team at the university, in parallel with the implementation of the PPM, contributed to the effective planning of future sessions as well as the documentation of the types of interventions that seemed to promote the construction of mathematical meanings by the pupils.

A.3.5 Questionnaire answers: ETL TE with MoPix

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching experiment? Explain in what sense, and how you coped with them.

One of ETL's main fields of interest has been the development of technology-enhanced innovative activities. Both during the phase of designing such activities and the implementation process, the Greek educational system's dominant characteristics have always been considered as an integral part of the institutional/cultural context of the teaching experiment.

The Greek educational system is considered to be highly centralised (Kontogiannopoulou - Polydorides, G. & Kynigos, C. 1993). The National Curriculum constitutes a pre-prescribed set of instructions for the teacher to follow, imposing uniformity in the educational practice and leaving no room for initiatives on behalf of the teacher or the school. Thus, the teacher is perceived as the technical implementer of the curriculum (Kynigos, 2004) having officially no right to implement his personal educational agenda or put into practice any teaching methods alternative to the traditional ones.

Although the use of technology is partially integrated in the National Curriculum, the *educational* use of technology is limited and depends exclusively on the teacher's will to integrate innovative practices in his teaching agenda. Even so, most of times, the use of technology is perceived –both by the teachers and the school administration- as a new, fancy way to deliver the defined by the curriculum content and not as means for expression and construction.

Inevitably, in this context, both students and teachers consider learning to be a merely individualistic procedure during which the teacher attempts to transmit knowledge, usually through problem solving activities, leaving no room for experimentation, meaning generation, personal expression, meaningful constructions and collaboration among peers. The implementation of innovative activities is extremely rare and bound to generate *perturbation* (Laborde, 2001) both in the school and the classroom-level.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching experiment? Explain in what sense, and how you coped with them.

The MoPiX Teaching Experiment was purposely designed to take into account potential problems the implementation process would generate with regard to the local context. The researcher who developed the Pedagogical Plan and made all the necessary arrangements to prepare the experimentation had been a teacher in the school in which the implementation took place for more than seven years and thus managed to take into consideration crucial elements of the context that would be likely to cause unanticipated problems.

Those elements concerned not only the turbulence an implementation of this kind would cause to the students' routine classroom practices and roles, but also practical issues such as the availability of the computer lab during certain school hours, the grant of permission from the school administration to implement innovative activities in the school and the other teachers' support and good will to provide the sufficient number of school hours for the implementation. The development of an open, flexible Pedagogical Plan that could be revised at any point by the researchers and the support the researchers would offered the students during the experimentation as well as a series of meetings with the lab administrator, the headmaster of the school and the rest of the teachers helped us to plan an experiment that would take into account most of the problems the local context could possibly create.

Moreover, both the researcher and the teacher researcher who conducted the experimentation had already carried out in the same school during the past year a pilot research for the Cruislet DDA with students of the same grade and age as the ones that participated in the MoPiX Teaching Experiment. This gave them a good idea of the potential problems that could possibly occur during the MoPiX experiment's implementation process.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching experiment? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

The implementation of innovative activities in rigid educational system such as the Greek one is bound to produce -at different levels and extent- some kind of perturbation. The fact that the ETL research team members are also active teachers who have deep knowledge of the global and institutional context in which the implementation took place, permitted us to minimize the effect the specific characteristics of the global context would possibly have on the implementation process and a-priori devise ways to by-pass potential problems.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching experiment? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

During the experimentations we encountered few technical problems (e.g at some point we were unable to retrieve data from the computers), most of which were solved by the researchers who participated in the experiment (one of them has a specialty in Informatics and is more a far experienced researcher that the teacher-researcher).

As far as the Pedagogical Plan was concerned, we had to revise the time schedule twice and modify some of the activities. The reason for that was the fact the students felt that they needed more time to spend working on specific activities. Since we didn't want to impose our own pace of work, disregard the difficulties our students encountered and move on to the next activities, we decided to prolong the corresponding phases and give them more time to work on their models. This could not be considered to be a local problem as if this was an obstacle deriving from the students' specific characteristics, but it could be recorded as an unexpected problem that had as a result to ask the teachers for more school hours to complete the experimentation.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

As it was mentioned before, the ETL researcher who designed the Pedagogical Plan and made all the necessary arrangements has also been a teacher in this school for many years. This fact enabled us to have full access to the school premises and equipment, communicate effectively with the administration and the teachers and eventually carry out the experimentation by-passing any obstacles the request of an official authorization would cause (consume time waiting for an approval, declare the exact time schedule that should be followed accurately).

The teacher's previous experience in the school permitted us to taken care of the context problems (particularly the local ones) in advance, during the planning phase, while any petty problems created during the implementation were handled on the spot mostly due to the fact that the teacher knew her ways around the school.

A.3.6 Questionnaire answers: IoE TE with MoPix

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

Within the UK educational system, the following issues are of main concern and interest:

- Entry to schools: In general, it is difficult to get into the schools in the UK. One needs to go through a long process of legal issues and coordination with schools. The ethical consideration is also an issue within the educational system and within the Institute of Education/LKL as well.
  This issue was resolved by using personal contacts to facilitate entry.
- Relationship to the National Curriculum, school schemes of work, examination pressures and the time available: while the system officially gives space for teachers to add enrichment activities, in practice the curriculum is very constrained and many teachers and schools are unwilling to deviate from standard schemes of work.
  We dealt with this in two ways. Firstly, our teaching sessions were scheduled outside the timetabled lessons for mathematics. Secondly, we negotiated with the teachers involved in order to design our pedagogical plan in a way that would have as much synergy with the prescribed curriculum as possible.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

- Curriculum match: during our discussion with the teacher, the issue of making the teaching experience relevant to the curriculum was dominant and we changed our plan to achieve that goal.
- The time available: The college and teachers were not prepared to allow us to use their scheduled teaching time for the teaching experiment. Students thus had to voluntarily give up some of their free time to be involved in the project. Finding a time slot to accommodate us was difficult, resulting in a smaller than anticipated number of participants.
- Students' priorities and study loads: This concern was raised by the teachers since the students were studying for a high staked examination in Advanced Level Mathematics. It was necessary to present teachers and students with explicit links with the standard curriculum and to present the project to students as an activity that would support their learning in preparation for their examinations - and look good on their CVs when applying to university.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

- The curriculum and examination system: These proved to be an ongoing issue, affecting student motivation and attendance. The college supported us by continuing to encourage students to attend.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

- Number of students and their priorities: Since participation in the teaching experience was voluntary, only seven students were involved. In some sessions attendance was reduced because of other events (e.g. attendance at interviews, preparation for examination) that took priority for individual students. This had the consequence that we were unable to rely on stable groupings of students. The collaborative working aspect of our pedagogical plan was thus reduced and adapted.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

Teachers were only involved at the planning stage. The teaching sequence was implemented by the researchers.

A.3.7 Questionnaire answers: IoE TE with MaLT

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

Within the UK educational system, the following issues are of main concern and interest:

- Entry to schools: In general, it is difficult to get into the schools in the UK. One needs to go through a long process of legal issues and coordination with schools. The ethical consideration is also an issue within the educational system and within the Institute of Education/LKL as well.
  This issue was resolved by using personal contacts to facilitate entry.
- Relationship to the National Curriculum, school schemes of work, examination pressures and the time available: while the system officially gives space for teachers to add enrichment activities, in practice the curriculum is very constrained and many teachers and schools are unwilling to deviate from standard schemes of work.
  There were three consequences of this for our planning. Firstly, some of our teaching sessions were scheduled outside the timetabled lessons for mathematics. Secondly, we negotiated with the teacher involved in order to design our pedagogical plan in a way that would have as much synergy as possible with the prescribed curriculum and with officially sanctioned approaches to teaching. Finally, the school chose a group of students for us to work with that was identified as 'low attaining'. Because these students were not expected to perform well in high-stakes examinations, allowing them to spend time on extra-curricular activities was not perceived by the school as so high risk.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

- The National Curriculum and school scheme of work: during our discussion with the teacher, the issue of making the teaching experience relevant to the curriculum was dominant. The plan had to be adapted to achieve that goal. We also had to schedule the teaching sessions to fit in with the school's planned order of topics.
- The time available: Given the small amount of time available within the school's scheme of work, we had to schedule all sessions within a two week period and, moreover, had to make use of additional after school sessions.
- Students' level of attainment: As indicated above, the group of students we were able to work with were identified as low attainers. We reduced our expectations of what students would be able to achieve based on the school's assessment of their attainment level.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

- The time available: as mentioned above, the main affect of the time was in putting the whole teaching experience only on 10 days with condensed schedule that added more pressure on students and affected their motivation and interests.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

- Students' level of attainment: this issue added more pressure on us as researchers since we were tightened, from on hand, by the time available and, from the other hand, by the schedule of activities to implement MaLT with the students. In addition, the students had no pre knowledge with LOGO.
- The school's computing facilities: The computer laboratory in which our MaLT sessions were scheduled was arranged in such a way as to discourage group work. Students were also used to working individually when in this room. This meant that they were unwilling to share computers and there was less collaboration and discussion than we would have wished.
- The usual didactic contract: Students were unused to working in groups and found it hard to do so productively. In order to help them to do so, we were supported by the class teacher and a student teacher.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

The teacher was the main figure of the teaching sequence in preparation and implementation stages. We coordinated with her from the beginning: the general plan, the time needed and the type of project as an outcome of the TE. Thus, the teacher's role did affect the implementation of the teaching sequence. One main contribution is her 'good' assessment of the pupils she is teaching which affected our plan and implementation of the TE.

A.3.8 Questionnaire answers: ITD TE with Alnuset

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

In the Italian secondary school, the teaching of algebra is devoted to the development of competencies of symbolic manipulation on the basis of a curriculum that is characterized by a quite rigid sequence of topics and notions (for example, the second-degree equation is approached after the first-degree equation, the inequalities are approached after the equalities…). The traditional curriculum of Algebra is justified by the characteristics of the techniques used in traditional practice. The instrumented techniques of Alnuset allow us to modify this rigid sequences of algebraic topics and notions. We have decided to exploit them to plan a PP characterized by innovative aspects at the curriculum level (for instance, we have approached the solution of the second-degree equation without introducing the formula to solve it). This change did not create problems because it was well supported by the instrumented technique of Alnuset.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

No problem, because the teacher was very interested and motivated to practice the innovative proposal characterizing our PP as well the specific school involved

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

No problem

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

No problem.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

The fact that the teacher was not a member of our research group did not affect the implementation of our TE. He appreciated very much the proposed PP and he managed well the activities with his students. At the end of the TE in a interview he expressed a very positive opinion on the performed activity. In this new school year he has expressed the intention to use Alnuset in his normal activity with his classes.

A.3.9 Questionnaire answers: ITD TE with Aplusix

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them

No problem in planning the TE since the DDA and the didactical goal were consistent with the curriculum

(**Planning – Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

The school we have involved in the TE is one of the three experimental low secondary school in Italy. With this school, that is well equipped from a technological point of view, we have a strong collaboration from several years. Hence, no problem

(**Implementation – Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

No problem

(**Implementation – Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

No problem

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

The teacher involved in our experimentation is a person with which we had already collaborated in the past . She actively participated in the design of the PP. Once defined with the teacher the arithmetic knowledge to be taught and the educational goals to be achieved, our research group designed a draft of PP exploiting the functions of Aplusix and, in particular, the modality of use of the tree representations. The teacher contributed in refining the draft version of the PP and she proposed to add some tasks. These tasks concerned the translation in natural language of arithmetic expressions expressed through both tree representations and the arithmetic linear representation

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A.3.10 Questionnaire answers: MeTAH TE with Aplusix

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

Our biggest concern was finding teachers who would be willing to experiment with Aplusix-tree. Initially, we wished to experiment with Grade 7 or Grade 8 classes. At this level, working with trees would be more appropriate and would contribute to the learning of algebra. Unfortunately, only the teachers who are members of our research team agreed to implement a scenario involving a tree representation, but they had Grade 9 and Grade 10 students. For this reason, we had to design activities with the aim of remediation to students' difficulties.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

Constraints in the schools, institutional or material.

At the institutional level, for example, in the school where C2 class is, all Grade 10 classes progress in the math curriculum in the same way (same order, same rhythm). For this reason, it was very difficult for the teacher to integrate the whole scenario into the common sequence.

Material constraints led to different implementations of the scenario. For example, in the Grade 9 class it is not possible to split the class into two groups. Therefore, all sessions were done with the whole class, 2 students per computer and the classroom orchestration was much more difficult than in the Grade 10 classes where the organisation allows working with a half-class.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

Although the 3 teachers are members of the research team, they were not involved in the design process of the scenario. This might actually have affected its implementation. In particular, the results observed in the C1 class in comparison with those observed in C2 class, lead us to question the way the teachers had appropriated the scenario and integrated it or not into their pedagogical activity. Recall that in C1 class, students' productions to activities that followed the introduction of the tree register (RT) show that they did not succeed in mastering this new representation and making connections with the usual one. Despite of the difficulties, the teacher kept moving ahead through the proposed activities. Two hypotheses allow explaining this behaviour:

- The tree representation is a mathematical object, which is not present in mathematics curriculum. The teacher herself might have felt uncomfortable with this content and had not succeeded the introductory session. Recall that in C2 class, the teacher actually preferred to ask one of the researchers to lead this session since he was not completely at ease with the new representation as well as with manipulating the trees in Aplusix. In retrospect, we realize that we could have under-estimated the fact that the teachers could be unfamiliar with the tree representation.

- The tree representation not being an institutional object, the scenario proposed activities on top of the regular institutional ones. Thus it seems that in the teacher's eyes, they have hindered the progress of the usual teaching sequence rather than brought a solution to students' difficulties.

A.3.11 Questionnaire answers: MeTAH TE with Alnuset

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

The design and implementation of the teaching sequence at stake is part of a Master thesis. The aim of the thesis was to analyse, both by inspection and empirically, an interactive learning environment, namely Alnuset. Thus, on the one hand, the teaching experiment aimed at validating hypotheses coming from the a priori analysis of the software. On the other hand, the scenario was elaborated by the Master student in agreement with the teacher of the experimental class. Therefore it was necessary to propose activities that could be easily integrated into the teacher's pedagogical sequence. Designing activities that would bring these two goals into harmony was the hardest problem in the planning the teaching intervention.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

Material constraints raised the greatest problems in the implementation of the scenario. The scenario was implemented during two sessions, and only during the first session the class could have been split in two groups. The second session was therefore much more difficult to manage. Moreover, the material arrangement of the classroom was not favourable to the whole class discussions. The students were obliged to move from the computer part of the classroom to another part with ordinary tables in order to be able to see the screen where Alnuset was projected by the teacher.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

A.3.12 Questionnaire answers: Unisi TE with Aplusix

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

In planning the experiments based on Aplusix, we cannot speak in terms of problems but in terms of points to be discussed with teachers. The two teachers involved asked us since the beginning to stay as far as possible close to the curriculum and to make explicit our envisaged links with the curriculum, in the cases in which the faced arguments did not appear so linked to it. We accepted teachers' requests and we tried to involve teachers in the planning of the experiment as much as possible.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

The answer to this question has been given above if considering the way the teachers (elements of the local context) affected the planning of the experiment.

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

No problem.

(**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

As far as concerns the classes, probably the greatest problem has been how to carry out the lesson when some pupils where not at school. This common situation affected the working-sessions, where students had to work in pair and on the contrary remained sometimes alone. Furthermore, when a pupil was absent for one or more lessons, once joined again the class, it was then difficult to reach her classmates' level. The usual way adopted by teachers, that is to make a short summary of what had happened in the previous lesson to help the students who was not present, seemed not to be effective for the type of activity of the experimentation. So, a different approach has been adopted. It was not up to the teacher 'to fill the gap', but were some students who working with the DDA (by means of a overhead projector) showed to the class the passages of the previous activity that they considered more significant.

(**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

Two teachers were involved in the experimentation: one was a member of the team since years, the other one collaborated with us for the first time. The teacher with experience in our team often influenced the working session, since she wanted in a way or in another to reach the objectives we had fixed; as a consequence, sometimes she intervenes too much. On the contrary, the other teacher limited her interventions, giving more time to her students to discuss collectively.

A.3.13 Questionnaire answers: Unisi TE with Casyopée

(**Planning - Global**) Which elements of the global context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

The teaching intervention was designed for grade 12 or 13 of Scientific High School. At the end of the grade 13 class of Scientific High School, students have to give a final year examination in mathematics. The content knowledge taught in that class has to be relevant to the final year examination: this is an expectation of both students and teachers. We tried to design activities so to meet this expectation.

(**Planning - Local**) Which elements of the local context raised the greatest problems or concerns in the planning of your teaching intervention? Explain in what sense, and how you coped with them.

N/A

(**Implementation - Global**) Which elements of the global context raised the greatest problems or concerns in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

N/A

> (**Implementation - Local**) Which elements of the local context raised the greatest problems in the actual implementation of your teaching intervention? Explain in what sense, and how you coped with them. Explain In particular issues which were unexpected.

The computer labs of some of the school involved in the experimentation were equipped with “low-performing” computers, which hindered running contemporarily Casyopée and Screen-Capturing Software (used for documenting students’ activity).

> (**Teacher**) Did (and possibly how) the fact that the teacher was/was not a member of the research team affect the implementation of the teaching intervention?

The teachers involved in the implementation of the teaching intervention were not members of our research teams, even if two of them had already collaborated with us in the past years. They agreed with the teaching intervention (structure, aims, activities…) and the underpinning principles, even for what concerned their specific role. Nevertheless they had not previous experience in managing classroom discussions as framed within the Theory of Semiotic Mediation (that is crucial in our intervention); the classroom discussions were not always exploited as completely and fruitfully as *a-priori* envisaged.

A.4 Syntheses of the Teaching Experiments

A.4.1 Synthesis of Didirem TE with Casyopée

Educational goals and students' achievements

Goals

The pedagogical plan aimed to help students to construct or enrich knowledge on two aspects:

1. meaning of functions as algebraic objects,
2. meaning of functions as means to model a co variation in geometric and algebraic settings.

More specifically :

as for the notion of function as an algebraic object, students should consolidate:

- the meaning of variable the distinction between variable and parameter
- the meaning of function of one variable with several registers of semiotic representation
- the fact that a same function may have several algebraic expressions

as for functions as means to model a co variation, students should develop:

- the ability to experiment and anticipate in a dynamic geometric situation
- the ability to modelling a geometric situation by a geometric then algebraic calculus
- the ability to interpret an algebraic result in the geometric context.

The plan proposed a succession of tasks exploiting the potential a priori offered by Casyopee for approaching and studying the notion of function, and especially:

- the role played by parameters for studying family of functions and generalizing.
- the role played by functions for solving problems arising from geometrical situations.

Specific importance was given to the construction of tasks where students can choose different variables for exploring functional dependencies, and to the connection between algebra and geometry. This connection is supported in Casyopee by geometric expressions that allow expressing magnitudes in a symbolic language mixing geometry and algebra.

The scenario was built around three main types of tasks:

- finding target quadratic functions by animating parameters (five different tasks according to the semiotic forms used for these functions):

Lesson 1: Introducing associated functions (a function g is associated to a function f if it is defined by a formula like g(x)=af(x)+b or f(ax+b) or similar)

Lesson 2: Target Functions (functions that can be graphed but whose expression is not known)

Lesson 3: Different expressions of quadratic functions

- creating a function as a model of a geometrical situation to solve a problem of relationships between areas,

Lesson 4: Introduction

Lesson 5: Application; dividing a rectangle into figures of fixed area

- creating a function as a model of a geometrical situation to solve an optimization problem.

Lesson 6: solving a problem of optimisation in geometric settings by way of algebraic modelling.

Achievements with respect to the a-priori envisaged ed. goals.

Goals related to the distinction between variable and parameter and the meaning of one variable function

In lesson 1, the teachers introduced the notion of associated function in close relationship to Casyopée's functionalities: a function f being entered (in the examples f was x² or sin(x), parameters help to define 'generic' associated functions. Then the problem was to find values of the parameters to match a "target" function that is to say).

Students learnt to create and graph functions then to create parameters and associated functions.
After that, in lesson 2, they had to learn how to animate parameters in order to find target functions by superposing the target's function and the associated function graphs.

Because Casyopée accepts only identifiers of objects already created, students sometimes had difficulties when creating associated functions, because of inappropriate creation of a parameter (for instance they had to create a parameter with the identifier h, but did not activate the choice of an identifier and finally created a parameter with the default identifier. After that, when they wanted to create an associated function involving the parameter h, they got an error message from Casyopée, warning that h was unknown). Nevertheless, after the first two sessions, students were quite familiar with these functionalities of Casyopee, and never confused parameters and the function variable.
In lesson 3, a "guess my function" game was proposed were students in a team had to immagine a function, an associated function and a target in order that another group solve it. Students actively participated, creating sophisticated targets.

In the geometric part of the experimentation (lessons 3 to 6), students used parameters to treat generic cases (for instance a rectangle of size a and b) and had no particular difficulties with geometric constructions and expressions involving parameters. Some exceptions occurred when students had to find a solution with parameters (see below lesson 6, team 2) and could only consider numerical cases. They managed without difficultu to build a figure and create a function involving parameters, but, to get a solution they had to apply a procedure that before they had used before only for numerical values.

This global achievement can be related to the clear statute of parameters in Casyopée, the facilities for animating them as well as to the careful and gradual introduction by the teachers.

Goals related to understanding that a same function may have several algebraic expressions

This was at stake in session 3. The target function was quadratic and students were given the associated function f(x)=a(x-d)^2+e g(x)=a[(x-k)^2+m] et p(x)=a(x-u)(x-v). They had to find values for parameters a, d, k, m, u, v. It was relatively easy for them to treat separately the associated functions f and g, because the task was close to what they had done in lessons 1 and 2 and they could make sense of the effect of animating parameters. However they could have gain time by making a connection between f and g. For instance remarking that e=a m, could have helped them to find m without animating parameters. No students did that. It was much more difficult for them from p. Animating parameters did not help them to make sense of u and v. They looked in vain for an effect of these parameters on the global form of the parabola without noticing the intersections with the x-axis. Actually there was a clear gap between the first two forms and the third. The first two were in continuity with the previous activity and animating the parameters produced displacements that students could make sense of by considering the curve globally. In contrast, making sense of the animation in the case of the factorisation requires considering 'locally' the intersection with the x-axis. Teachers had to recall students more or less individually the characteristics of the form and the relationship with the zeros. The recording shows that students did not immediately understand their point.

Students were also asked to check their answers by expanding the expressions they found for f, g and p. It was hard for most students. Students that did not find the same expansion because of mistakes wondered whether something was wrong or not. However, when they corrected and found the same expansion, they were relieved and recognized the uniqueness of the expansion.

Students were also asked to use the expression they found, to look for the extremum of the function. It was expected that they recognize e as an extremum both from the curve and because a(x-d)^2+e is the sum of the constant e and of an expression of constant sign. Few succeded without help.

Students had to do this twice for two different quadratic functions. For the second function, the expanded expression was given and it was not especially suggested that students use associated function and parameter animation. They nevertheless used these. Even the parameter a, which in the three forms is the coefficient of x^2 in the expansion was found by animation. Animating was not easy because students had to adjust the step in order to get non-integer values of the parameters. This second function had no factored form and students recognised more or less easily this by noting that the curve does not intersect the x-axis, while animating the parameters u, v, the associated curve always did.

This report does not mean an underachievement with relation to the goal of understanding the several equivalent expressions of a function. This shows that, although these students learnt algebra before and were relatively high achievers, their algebraic knowledge was still weak both with regard to manipulation and to understanding. Actually this insufficient knowledge was challenged by the tasks and clearly they progressed with regard to "completing the square" forms as well as to the expansion (uniqueness). This progress is less visible with regard to factorisation. Teachers' dialogs with students can be described as 'strong mediations' and question the 'a-didacticity' of the situation. Up to what point could Casyopée's feedback make students give up with the global point of view and reflect on the properties of the factored form? Up to what point was this mediation effective?

Casyopée played the role of an enhanced grapher, more than of a symbolic environment. Students could have obtained the expansions and the factorisations directly via the calculate menu of Casyopée. Few did it and only for double-checking. Teachers did not encourage them to use this menu, implicitly recognising that 'by hand' calculation was important at this step.

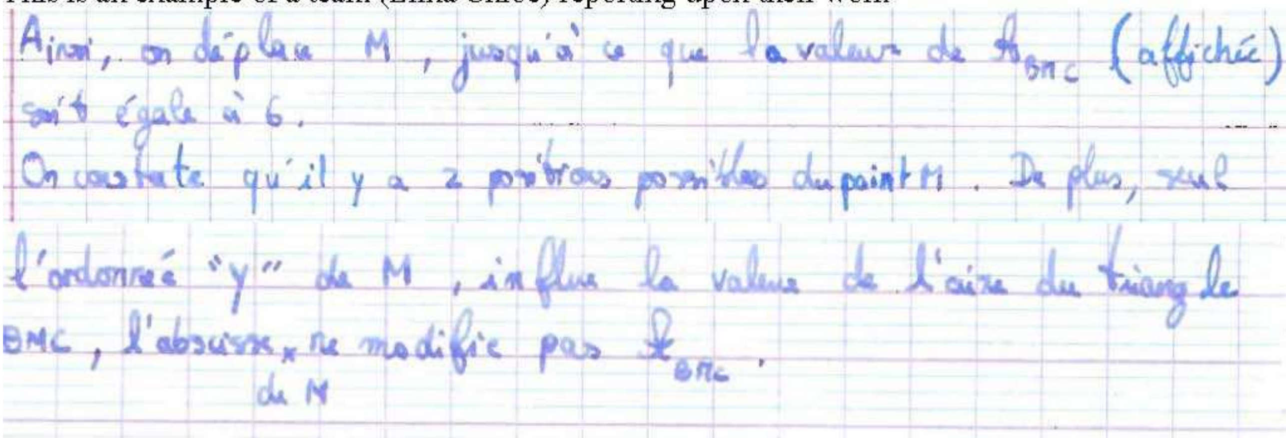
Goals related to the meaning of a variable and of function of one variable

Whilst in the first three sessions, a function was for the students a familiar object linked to an algebraic expression, lessons 4 and 5 proposed a wider approach based upon the dependency between measures. Lesson 4 was an introduction to problems about measures in geometry and to the associated Casyopée' functionalities. In lesson 5, a free point M in the plane was defined and students had to build a function modelling the dependency between M and an area involving this point in order to find the positions of M that give a specific value to the area. The area was chosen in order that it depended on the y coordinate of M, but not on the x. Thus students had many choices for an independent variable built with M, but only variables like yM or yM-yU or yU-yM (U being a fixed point) were adequate for building a function. It was expected that students would understand this by interpreting the Casyopée's various feedback.

In the experimentation, students were prepared by a study of the co-variation: by dragging M, they could approach a position where the area had a required value. Some noticed that moving M 'horizontally' did not change the area.

Nevertheless there was much hesitation when choosing a variable, students moving back and forth between xM and yM and asking the teacher how to understand Casyopée's feedback, before finally choosing yM. It seems however that most understood better the meaning of a variable and a function of a variable thanks to the task and to Casyopée.

This is an example of a team (Elina Chloé) reporting upon their work


Ainsi, on déplace M, jusqu'à ce que la valeur de A_{BMC} (affichée) soit égale à 6.
On constate qu'il y a 2 positions possibles du point M. De plus, seul l'ordonnée "y" de M, influe la valeur de l'aire du triangle BMC, l'abscisse x du M ne modifie pas A_{BMC}.

They say that they moved M until reaching the expected value of the area. They say that they found two positions, which is not entirely correct because all positions on two parallels to the x-axis are solutions. This means that dynamic exploration was not sufficient in itself. After that they recognized that the area depends on yM and not on xM, as a result of Casyopée's feedback when choosing a variable and creating the variable.

Thus both the exploration in Casyopée's GD and Casyopée's specific capabilities for modelling contributed in recognizing the functional dependency at stake.

They also commented upon how Casyopée helped them and upon their difficulties: Their comments about Cayopée's help confirm the above analysis and their difficulties show that "finding that y alone has an effect upon the area" was really the big challenge of the situation.

AIDES APPORTÉES : Mettre en relation la géométrie et les fonctions, grâce à la visualisat° :
Mieux comprendre les calculs d'aires, les données à choisir....
Travailler d'1 manière + ludique.
DIFFICULTÉS : Comprendre la consigne, l'objectif de la question posée et les méthodes afin d'y répondre.
Visualiser, voir, interpréter ce qui est demandé
Trouver quel seul y influe l'aire de BMC, d'où le choix de yM comme variable des fonctions.

The following report by another team brings also evidence of the role of Casyopée's specific capabilities for modelling in understanding the dependency upon yM.

Suite à plusieurs expériences sur Casyopée avec différents choix de variables, nous avons remarqué que seule la variable yM nous donnait un résultat (car xM $S =]-\infty ; +\infty[$).

In contrast Kévin did a throughout exploration and found the sets of solutions without algebraic modelling.

je fais donc OA×OB. puis je crée un point libre nommée M. Je trace ensuite la perpendiculaire a BC passant par M. ~~je crée donc le~~ calcul l'intersection de la droite et de BC s'appelle H. puis je peux calculer l'air de BMC avec le calcule $\frac{BC \times HM}{2}$.
ainsi par tatonement j'obtiendrait comme solution tout les point de la droite passant par le point K (0 ; 2) et parallele a l'axe des abscices.* De même les point de la droite symétrique a celle ci par raport a BC sont aussi solution.

As a last example showing diversity among students, Charlotte's report brings no evidence of a focus upon the independent variable:

Ensuite, nous avons exprimé la fonction géométrique égale à l'aire du triangle BHC ; d'expression algébrique $f(x) = \frac{a\sqrt{x^2}}{2}$
Une fois ces fonctions créées, nous avons effectué l'équation suivante $f(x) = g(x)$ pour pouvoir trouver la valeur de x quand l'aire du triangle BHC est égale au tiers .

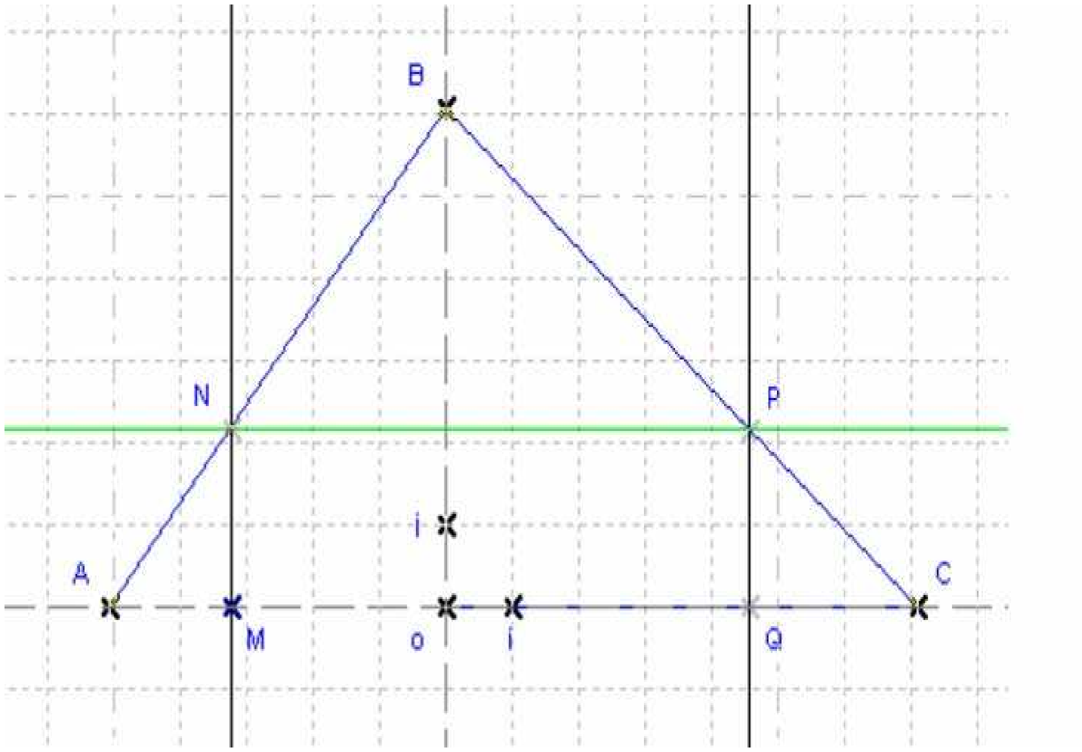
Goals relative to

> ***the ability to experiment and anticipate in a dynamic geometric situation***
>
> ***the ability to model a geometric situation***
>
> ***the ability to interpret an algebraic result in the geometric context.***

Lessons 4, 5 and 6 were conceived for these goals. Lesson 4 was an introduction to problems about measures in geometry and to the associated Casyopée' capabilities. In lesson 5, students had for the first time to solve a problem by modelling algebraically a dependency. As shown above, the main stake was the idea of a variable that could be relevant to model the position of a free point for a given problem. They were introduced to a method: creating a geometric calculation representing the area at stake, choosing an independent variable, creating the corresponding function, solving the problem in the symbolic window and visualizing the solution.

In lesson 6, no method was indicated and students could choose freely a construction of the rectangle (especially choosing the free point), a variable and a method. The method followed in lesson 5 was nevertheless likely to influence students. Actually what was at stake was students' understanding of this method and their ability to put it into operation for a new problem (finding a position optimising an area, whereas lesson 5 was about finding a position for a given value of an area). We will analyse this lesson 6 relatively to students' work.

The problem that students had to solve was the following. a, b, c, being 3 positive parameters. Three points are defined: A(-a,0); B(0,b); C(c,0). Consider a rectangle MNPQ (with M on [OA] ; Q on [OC] ; N on [AB] and P on [BC]) . Find the rectangle of maximum area.



The students had to create the rectangle, to solve the problem, to put in writing their solution, to write a research report, and to visualize the solution in the geometric window.

The observation of students' work suggests that they appropriated diversely the method. This diversity among students can be appreciated with regard to three poles:

1. the study of the covariation in the geometric window for experimenting at the beginning and for checking the solution found symbolically at the end.

2. the study of the functional dependency on a curve in the symbolic window after expressing the covariation in an algebraic form.

3. the study of the functional dependency by algebraic means (finding a maximum symbolically)

The indicators of diversity are the importance that students gave to each pole in their work and the help that they needed from the teacher when working in each pole. The figure below gives a comparison between three teams and illustrates the variations.

Team 1 (green arrows) did a geometrical exploration of the covariation and concluded that the optimal position of M is the midpoint of [oA]. The teacher had then to prompt them for a proof. They choose the variable –xM that corresponds to the distance between o and M. One can think that they preferred to choose a variable defined on a positive interval. They created and graphed the function, but they needed the teacher's help to recognize a parabola. They knew how to calculate the extremum of a quadratic function, but did not succeed in this case because of the parameters. Then they calculated with the numerical values of the parameters and did not come back to the geometrical window. The teacher had to prompt them to interpret their result as the midpoint of the segment, consistent with what they found by exploration.

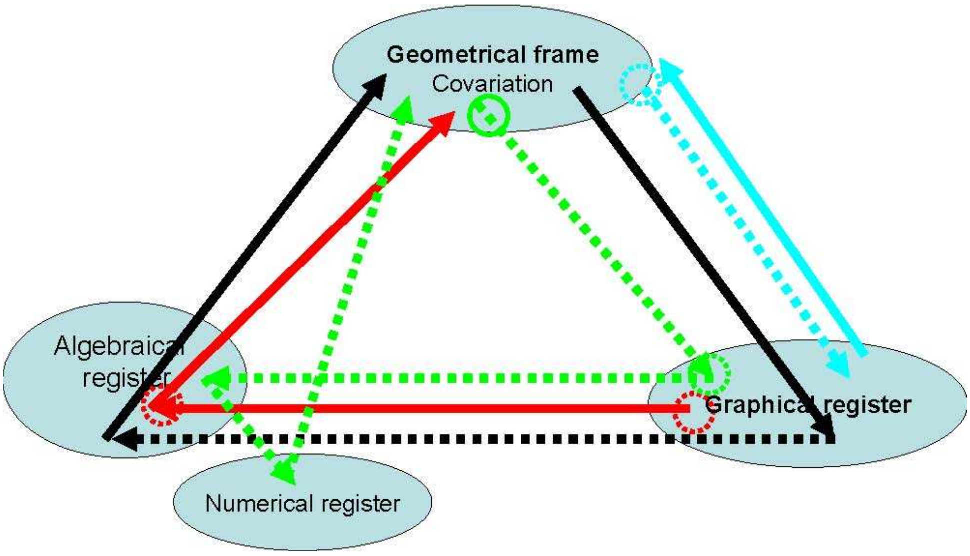
Team 2 (red arrows) did no preliminary experimentation. They asked the teacher to help them to choose an independent variable and directly created the function. After that they again needed teacher's help to study the covariation on the curve. They easily recognised a parabola and a maximum on the parabola, but were confused by the parameters when trying to express algebraically the position and the teacher had to help them to find a symbolic solution. After that, they went to the covariation in the geometric window and were happy that the solution they found algebraically was really a maximum in the geometric window.

Team 3 (black arrows), like team 1, did a geometrical exploration of the covariation and conjectured that the optimal position of M is the midpoint of [oA] and the teacher had to prompt them for a proof. He had also to help them to choose a variable and to create the function. They recognized the curve as a parabola and tried to read the coordinates of the maximum on the graph. The teacher had to help them to remember how to calculate the coordinates of the extremum. After that, like team 2 they were happy to see that the symbolic solution matched the result of the geometrical exploration.

Team 4 (blue arrows), was helped by the teacher to create the geometrical computation MN cross QM. Teacher also helped the team (only one student) to create the variable xM , the function and its graph. The student recognized the parabola but she can't see the maximum because of the zoom. Another student helped her to use the zoom. Then, she wanted to read the coordinates of the maximum on the graph with the cursor. She manipulates M in the geometrical frame in order to put the plot of the graph on the maximum. Finally, no proof on the algebraic frame was done.

With respect to the above goals, these reports show a global achievement, but also diversity among the three teams. Team 2 is not comfortable with experimenting and anticipating. They prefer to tackle the problem symbolically as soon as possible. In contrast team 1 and 3 favour experimentation and show some reluctance to model algebraically this geometric situation. Interpretation is difficult for team 1. More analysis is in progress, considering other students' work.

Green : team 1 ; Red : team 2 Black : team 3 ; Blue : team 4



Continuous arrows or circles when the team was working alone and discontinuous ones when the teacher had to help the team.

2. Specification of the evidence supporting the claimed achievements

The Didirem team privileges an "internal" assessment to claim about students' achievement and about the role of Casyopée and of the pedagogical plan. This is clearly different from "external" assessment, that is to say comparison between experimental and control groups.

“Internal” assessment is based upon a comparison between the a-priori analysis of the situation (analysis of the milieu and of its feedback, analysis of the support brought by the software and by teacher’s mediation…) with the actual realization.

“Internal” assessment has proved much efficient to understand the role of the multiple factors in a learning situation especially with technology, whilst the evidence of achievements given by “external” assessment does not specify the role of the factors. In the case of our experiment, given the complexity of the learning situation considered in the research questions, the multiplicity of the factors involved and the relatively small scale of the experiment, a comparative study would have brought little new understanding. In addition, it is the first real research experiment about Casyopée use and an internal assessment is better for getting insight into the effects of this use.

This choice of an “internal” assessment brought us to gather data especially by recording the interactions in the classroom and students’ interaction with Casyopée. The analysis is in progress and the above section (*Achievements with respect to the a-priori envisaged ed. goals.)* can be considered as a first step and shows again the productivity of “internal” assessment.

3. Relationship with the CRQ and SRQ, and the ITF

With regard to the ITF and the C(S)RQ, the above analysis confirms the importance of semiotic phenomenon brought about by the experimentation. More work is needed to articulate a semiotic approach in terms of registers (Duval) that seem useful “inside” respectively a geometric frame (exploring co variations and dependences, and building functions in the GD window) and a symbolic frame (considering the algebraic expression of a function, its graph, the associated algebraic methods…), and an approach in terms of coordination of frames or settings (Douady).

Synthesis with regard to Casyopée’s potentialities and with regard to CRQ and SRQ are also in progress. In addition to semiotic issues, they will bring to the forth important issues like students’ instrumental genesis of Casyopée and the role of the teachers, putting at stake other elements of the ITF like instrumentation, the didactic-ergonomic approach of teachers’ practices and the TAD. The above analysis already gives insights into the interest of a graduated approach of Casyopée’s functionalities in relationship with mathematical tasks for students’ genesis. It also brings evidence that teachers had to take a lot of decisions often not predictable in a pedagogical plan, while preparing concretely the lessons, and also during the lessons themselves. More work will be necessary to identify clearly and to make sense of these decisions.

A.4.2 Synthesis of Didirem TE with Cruislet

1. About students' achievements

1a. Achievements with respect to the a-priori envisaged ed. goals.

Educational Goals

Use Cruislet’s potential for students working on 3D realistic problems enriching the meaning they give to vectors through the use of representations non standard school level and the meaning they give to curves such as circles, spirals or helix through the local generation of these.

These goals were only partially achieved because, in the two experiments, time was short (3 and 2 sessions). Students actually encountered the problems. Some were able to solve them completely while others found the software difficult to use and the tasks very demanding. Time was too short to really intrumentalize Cruislet. Passing to a paper pencil 2D representation to solve a problem of 3D displacement, coordinating Cartesian coordinates and polar representations of vectors as well as working on LOGO programs were real obstacles for some.

1b. Specification of the evidence supporting the claimed achievements

Data collected

- The first experiment:

– Video for the 9 (3x3) sessions observed

– screen captures for 12 students during the individual or group work sessions observed

– audio-recording for 5-6 groups for the same sessions

– successive versions of scenarios, comments by teachers, students' documents

- The second experiment:

– videos for the 2 sessions observed

– screen capture for 4 (2x2) groups of students

– students' documents

Elements informing answers

- Successive changes introduced in the design of the sessions by the teachers
- Analysis of videos and teacher mediations
- Analysis of screen captures on specific tasks:

– the Athens-Sparta trip in sessions 2 and 3 (exp1)

– the horizontal triangular flight and its vertical adaptation in session 3 (exp1)

– the landing near Mount Olymp (exp2)

– the Athens-Corinth flight (exp2)

– the adaptation of the Logo program for an acrobatic flight (exp2)

2. A section discussing the relation between students' achievements and the use of the DDA in the context of the PP. In this section the issue of 'representation' should be addressed, according the different theoretical approaches that each team adopt

a) Characteristics of the DDA

Cruislet's attractiveness and affordances for multi-representation have a counterpart: the complexity is very high. There are three ways of navigating associated to three different representations: with the mouse, by piloting avatars first by hand, then by LOGO programming. After students learnt to navigate with the mouse, they moved to piloting avatars, but then they could not navigate with the mouse anymore and they were often lost on the chart. That is why they had often to remove and recreate avatars. Then the avatar panel is very complex with several entry boxes some moving the avatar in different ways, and other related to the view (camera properties). It seems that many students do not really master this panel and the associated representations. Exporting to LOGO is done via the same boxes: only a check box controls two very different behaviours and representations in the DDA, piloting an avatar or writing commands, that after execution, will produce the avatar's move. The LOGO panel often confused students because they have no experience of programming,. They for instance had difficulties to insert exported commands at the right place as well as to edit consistently the program.

b) Educational goals

After reflection, when preparing the sessions, Cruislet representation of vectors did not seem to us a major feature. Thus our Mathematical goals were in relationship with 3D coordinates and trigonometry. These notions are not easy for students and problems with the interface were often mixed with mathematical difficulties, for instance understanding the difference between setpos and setdir had to do with distinguishing points and translations. Difficulties were also a consequence of insufficient ability to represent mentally the third dimension and of lack of method for solving problems in 3D. For instance a student positioned an avatar low above Sparta and wanted to go back to Athena, simply by choosing this town in the list. He repeatedly got the message "Avatar cannot go to this position" because there is a mountain very close to Sparta. He understood that he had to increase the altitude, which he did by trial and error up to 4000 meters, without thinking to go up above Sparta sufficiently high before taking the direction of Athens.

More generally students did not try alone to represent a problem like going from Athens to Sparta in the 2D vertical plane passing by these two towns. After teachers induced them towards this representation they had difficulty to activate their trigonometric knowledge (using atan to find vertical angles).

c) Modalities of use

The tasks we prepared in the pedagogical plan seem a posteriori well adapted for the goals. Nevertheless most students could not achieve them alone, in spite of Cruislet's representational capabilities, and one can be doubtful about what they actually learnt. A minority of students were more active and would deserve further analysis.

Less ambitious tasks (free exploration, trips without constraints…) could have helped students' appropriation of Cruislet' representations, and students could have achieved them alone, but they would not have put actual mathematical knowledge at stake, which is not really acceptable in the French institutional context. More simple tasks would also have been possible by overlooking the geographical background, for instance by making an avatar fly along a horizontal geometrical figure. More ore less consciously, we did not dwell on such tasks, because we thought that they do not exploit the Cruislet's representational potentialities.

Certainly, a more careful preparation taking into account the instrumental needs of the tasks we prepared would have brought better results. It would have required at least doubling the number of sessions. This again points towards the difficult ecology of this piece of software in the French institutional context.

3. A section addressing the issue of the relationship between what envisaged when planning the PPs and the actual results of the TE

Regarding the first experiment

- Analysis shows that, in spite of the interest shown by the students for working with the software, instrumentalization of the different representations and the coordination between these required by the piloting of avatars took more time than anticipated:
  - piloting avatars using directions
  - coordinating map and avatar use
  - coordinating direct and programmed piloting
- Analysis of the first experiment also attracts our attention:
  - on the mathematical requirements of the tasks proposed to students in the first phase of instrumentalization (the risk of cognitive overload was certainly under-estimated in the design of the tasks)
  - on the influence of institutional norms and their influence on teachers' decisions even if this specific context of TPE seeming less constrained
  - on the limited opportunity that students have for making sense of the semiotic affordances of Cruislet by the way of a-didactic adaptive processes, in spite of very interesting opportunities

Regarding the second experiment

- Two sessions suggesting:
  - that some main Cruislet features are quickly accessible to grade 9 students
  - the influence on these positive outcomes of the changes introduced in the scenario in terms of tasks and of the tight interaction between the group and collective work along the session
- But also, the same difficulties met like at 11th grade with the design of a flight under constraints requiring the use of some trigonometry and Pythagoras theorem
- The impossibility to getting a precise idea of what has been really learnt.

A.4.3 Synthesis of ETL TE with Cruislet (1/2)

1. Students' achievements

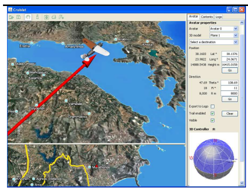
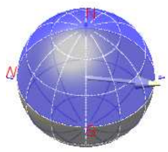
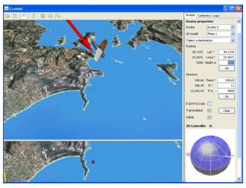
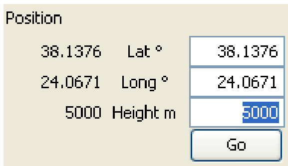
Our findings during the analysis process indicate that most of our educational goals were achieved at the epistemological, cognitive, social or instrumental level. In our perspective we endeavor to study students' construction of meanings, rather than indicate the achievement of the educational goals. According to our preliminary findings we separate students' meanings into the following categories, which are not based on the a-priori envisaged educational goals but are consistent with them.

1. *Geographical and spherical coordinate systems*

Students’ interaction with the Cruislet environment engaged them with concepts related to the two systems of reference used to navigate in 3d space, geographical and spherical coordinates, as well as with the relationship between them. In particular, regarding geographical coordinates, students preferred to use them to specify a specific position, in contrast to spherical coordinates where students used them to make displacements in space, independently of the destination place. An interesting issue is that students confronted latitude and longitude in a different way as they manipulated height in order to specify a position in space.

Most of the teams used spherical coordinates to navigate in space and in particular they used the 3d controller representation. Although they were not accustomed to this system of reference, they manipulated the 3d controller and through this they explored the notion of vector as the displacement and associated airplanes’ displacement with the variation in geographical coordinates. In this way, students explored vectors’ properties as they constructed links between geographical coordinates (the variables of the vector of displacement) and the spherical coordinates.

As an example of this we use the following sequence of students’ interaction with the environment, where they utilize both spherical and geographical coordinates to specify a position in space.

Cruislet environment	**Representation**	**Students’ actions**
		Manipulate 3d controller in order to specify direction of displacement.
		Change the height in 5000 meters and displace the airplane by pressing the ‘Go’ button.

The sequence of students’ actions indicates that they endeavour to associate the displacement in 3d space through the use of both systems of reference. Initially they use the 3d controller representation (spherical coordinates) and in this way they specify a specific point on the map as the geographical coordinates change simultaneously. Their second action includes the setting of one of the geographical coordinates as they want to place the airplane at a specific height on the map. In this case students utilised both Cruislet functionalities and the representations provided, as they attempted to combine the two systems of reference to displace the airplane.

2. *Function as covariation*

While students were interacting with the Cruislet environment several meanings emerged regarding the concept of function. We chose to categorize these meanings according to distinct concepts that rely upon the concept of function. In particular, there are three major categories:

- *Domain of numbers*: Students navigating the airplane in the 3d map of Greece realized that the domain of the geographical coordinates is actually a closed group. The investigation of the range of the geographical borders as the domain of the function became the subject of study and exploration through the use of the DDA functionalities. Students experimented by giving several values to geographical or/and spherical coordinates and by this defined the range of the coordinates' values. An interesting issue is that the provided representations helped students to realize that the domain of numbers in the airplane that was displacing according to the other one, were strongly dependent on the other airplanes' domain of numbers. In the following episode they realize that the spy doesn't follow them when they fly at low height.

  S1 There are some times that he (meaning the other airplane) can't follow us.
  R Where? When?
  S1 When I'm getting into the sea.

- *Function as covariation*: Initially most of the students expressed the dependency of airplanes' positions using verbal descriptions, such as behind, front, left, etc. as they were visualizing the result of airplane's displacements. When students experimented by giving several values to coordinates, they successfully found the dependent relation of the function in each coordinate and in this way they confronted function as a local dependency. It is interesting to mention that students separated latitude and longitude coordinates and the height coordinate as they were trying to decode the hidden functional relationship between the airplanes' height coordinates. In particular, they didn't encounter difficulties in decoding latitude and longitude relationship in contrast to their attempts to find the height dependency. An interesting point is that students used the height relationship as the rate of change of the function, as we can see in the following episode.

  S2 *When we go up 1000, he goes up 1000.*
  R Do you mean that if we go from 7000 to 8000 he goes from… lets say 2500 to 3500.
  S2 He is at… 3000. No. Give me a moment. At 8000 he was at 5500. At 7000 he was at 4500. At 5000 he is as 2500. And then….
  S1 We could do the division to see the rate.

- *Inverse function*: Almost all of the teams failed to find out that in a particular case the inverse function was needed to end the game, meaning to get the second airplane to a particular city (Thessaloniki). Most of the students were helped by the way the dependency in airplanes' position was represented on the map
  For instance, team 9 initially used the coordinates of Thessaloniki and displaced the first airplane. Seeing the result displayed on the screen, they realized that they had to use the inverse function to move the second airplane to Thessaloniki. In the following episode, student S1 advises S2 to use the inverse function, regarding it as a reverse process. Students have already find out the hidden function so in order to define this particular displacement they have to take into account the functional relationship between the geographical coordinates. The hidden function was (lat-0.1, long-0.05,

height-2500). Consequently, in order to navigate the second airplane to Thessaloniki, they had to change the geographical coordination according to the function.

S1 In order to get him to 500 meters we have to be at 3000 meters. (referring to height function)
S2 Yes. But he didn't disappear.
S1 You have made a little mistake though. Did you add 0.1 and 0.05? (referring to latitude and longitude functions retrospectively)
S2 What do I have to do?
S1 You have to … these are the coordinates that you must insert here.
S2 Yes.
M1 You have to add 0.05 and 0.1 in lat and long.

Although students used the representations provided to find the inverse function, when they had to express it in a symbolic way (i.e. when they were interfering in the Logo code), they came up against problems and they were confused.

3. *Combining mathematical and geospatial concepts*

In the Cruislet environment, mathematical concepts are integrated with geospatial representations and information, providing opportunities for processes of mathematisation of geographical space. Several times, students associated the two systems of reference with the geographical information. For instance, they correlated height in geographical coordinates with the mountains of Greece or the r in spherical coordinates with the borders of Greece as shown in the following episode.

S In r, fi and theta we have restrictions also.

R We have restriction in r, fi, theta? Tell me.

S Because we can't go outside the map of Greece.

…… (conversation about what r, fi, theta represent)

R Nice. And why do we have restrictions there? What is the relation between Greece and r, fi, theta?

S We only have the map of Greece, we can't go out of Greece.

R What values we can use, let's say on theta?

S Hmm… theta and fi can take any value we want to. Just the other, r can't be very large, because it's how far it will go and we can't get out of the map of Greece.

R So, the restriction is only for r?

S Yes.

Also in another case, they organized a flight trip according to the goal of the PP where they had been asked to construct a game for their schoolmates. Motivating by this goal they tried to make it as complex as possible. They defined the route of the flight so that it formed a triangle. The vertices of the triangle were three major cities of Greece. The goal of the game was to construct the triangle whose vertices were the midpoints of the sides of the first triangle.

2. Relation between students' achievements and the use of the DDA in the context of the PP

The computational environment supported students' experimentation at several levels regarding representations as it provides dynamic visual means that support immediate visualization of multiple linked representations. Students used the provided representations to construct meanings amongst themselves as any action carried on a specific representation provided immediate change and feedback in all representations. For instance, a displacement of the airplane using the geographical coordinates, provided students with visual feedback on the map and at the same time a numeric feedback at the spherical coordinates, as they were changed according to the airplane's displacement. As a result students constructed meanings about the relationship between coordinate systems and the displacement of an entity in 3d space. Apart from the visual or numeric representations, the Logo programming language provided opportunities for students to express navigational concepts in a symbolic way. In this way students associated symbolic representations with the visual one as they were running Logo commands and they were provided with the result on the map.

3. The relationship between what envisaged when planning the PPs and the actual results of the TE

<u>Time schedule:</u> The implementation of the TE changed some of the planned sessions of the PP. For instance we needed to extend the familiarization with the computational environment phase, as we realized that students needed more time to get accustomed to the representations provided and to get familiar with the concepts embedded in the environment and the actions in them. According to this change, students got familiar with all the representations (visual, symbolic, numeric) of the environment during 8 hours, before the implementation of the "Guess my flight" activity. Additionally due to context problems there was an extension of the period of implementation of the TE.

<u>Cognitive goals:</u> Most of the teams had difficulty in understanding issues regarding geographical coordinates. A possible explanation might be that students confronted difficulties with decimal numbers and the addition or subtraction between them. This was really surprising as we didn't expect students at that level to confront difficulties with decimal numbers.

A.4.4 Synthesis of ETL TE with Cruislet (2/2)

1. Students' achievements

The context of geographical space and the navigation within it provided students with the opportunity to construct mathematical meanings concerning the concept of function.

4. *Domain of the function*

   Students navigating an airplane in the 3d map of Greece realized that the domain of the geographical coordinates is actually a closed group. The 3d map of Greece is a geographical coordinate system with certain borders. The investigation of the range of the geographical borders became the subject of study and exploration through the use of the DDA functionalities. In particular, students exploit the two different systems of reference and, approaching the values of the geographical coordinates, they define the range of the lat – long values. This certain range of values has been considered to be the domain of the functions according to which the displacements of the airplanes are relative.

5. *Function as covariation*

Students exploiting the two coordinates system of reference constructed meanings of the function as covariation. An interesting example was the cases of the variation of the height of the airplane every time they push the button ‘go’ of the direction. In particular, students defining the vector of a vertical upward displacement observed that height was the only element that changed in the position of the displacement. Through a number of identical displacements students identified and expressed verbally, symbolically and graphically the dependency of the direction functionality and the height of the airplane. Students’ reasoning: “*the more times we push the button GO the higher the airplane goes”*, suggests that students developed a covariational reasoning ability similar to the second level proposed by Carlson et al 2001 of how the variables are changing with respect to each other. Moreover, the retrospective symbolic type that was developed by the students, h2= h1+1000, shows that they realized that the rate of change of the height is constant.

Students developed covariational reasoning abilities as they watch the flight of an out of order plane. Through the procedure of logo named fly1 (which was a black box for the students)they define the position of the plane(e.g latcorrect-intependent variable) and they see where it actually goes(latwrong-dependent variable). Trying to find out the hidden function, students exhibited behaviors that suggested they were able to coordinate changes in the direction and the amount of change of the dependent variable in tandem with an imagined change of the independent variable. We also noticed that students had difficulties in using the same reasoning patterns when attempting to construct a graph.

6. *Inverse function*

Students’ interacting with the software considered the inverse function as the reverse process in a way that the old outputs could became new inputs. Particularly, as students know the hidden function they were asked now to navigate the out of order airplane towards a particular city (Rhodos). The student made assumptions concerning the values they have to input. Their wrong guesses caused misdirection of the airplane. The immediate feedback provided by the DDA encouraged students to think of the concept of inverse function as a process that may be reversed (Carlson 1996).

7. *Identity function*

Students studying dynamic functional relationships, consider how an image of two variables changing simultaneously. During a particular phase of the PP the activity required them to reformulate the hidden function in a way that the dependent variable has the same values as the independent. Students could express easily the concept of identity function either verbally, symbolically or graphically.

8. *Proportional reasoning*

Students exploring the hidden function conceived the functional relationship of the geographical coordinates mainly as proportional. The hidden function involved the variation of the height of the airplane was height^2. Initially, students considered that the height had been multiplied by 200, misled by their experimentation with the value of 200 which gave the output 40000. After a number of explorations with different values they realised that “the height has been multiplied by its own value”. This alteration of their initial consideration was caused through their interaction with the DDA, although the proportional reasoning is deeply embedded in their thinking. De Bock et al., (1998) stressed a deep-rooted tendency in 12–16-year old students to apply proportional reasoning ‘anywhere’. Students’ tendency to proportional reasoning was also explicit in the graphs that they produced in order to express graphically the functional relationships of the geographical coordinates.

9. *Verbal, symbolic and graphical representations*
   During their experimentation with the Cruislet environment students constructed meanings about the concept of function. Students expressed the provided representations verbally, symbolically and graphically. Analyzing students’ constructions, several difficulties occurred. For instance, students could verbally express meanings which emerged from their interaction with the available representation, but they confronted difficulties to express them symbolically in a mathematical way.
10. *The concept of limit*
   Students navigating the airplane in 3d space developed an interesting intuitive approach to the concept of limit. Specifically, while they had to approach a specific point on the map, they used the spherical coordinate system of reference by gradually reducing the measure R of the vector of displacement. The students’ strategy seemed to be closely related to their idea of the concept of limit “I approach something as near as possible”, “I had to reduce the step…”.

2. Relation between students’ achievements and the use of the DDA in the context of the PP

Students exploiting Cruislet functionalities engaged in a number of activities involved in the PP. In particular, students had to find out the hidden functional relationships between the geographical coordinates of the different airplanes’ positions using the provided systems of reference (spherical and geographical coordinates) either in the visual and numerical context and/or in the Logo context. In the case of the domain of the function, students define the borders of the 3d map of Greece using both systems of reference as they had to navigate in the particular geographical space.

The development of students’ ideas about the function as covariation was supported by the variation of the Logo procedures and the simultaneous results on the displacements of airplanes. By experimenting with the variation of the geographical coordinate of the position of the airplane they explored the hidden functional relationships and developed covariation reasoning abilities. Students expressed these ideas in several ways, such as verbally, graphically and symbolically.

3. The relationship between what envisaged when planning the PPs and the actual results of the TE

Based upon our PP and the results of the TE, students constructed several meanings around mathematical concepts. In particular, students:

- developed covariational reasoning abilities while they were experimenting by giving several values to coordinates (either geographical or spherical) and they observed the result displayed on the screen,
- constructed meanings around the concept of function such as the inverse or identity function,
- used their intuition to construct meanings about the concept of limit,
- created verbal and graphical representations of functions,
- became familiarized with spatial concepts.

A.4.5 Synthesis of ETL TE with MaLT

SECTION A: Students' achievements

Throughout the implementation of ETL's pedagogic plan 13-year-old pupils were engaged in exploring the mathematical nature of angles while controlling and measuring the behaviours of geometrical objects in the simulated 3D space of MaLT. In MaLT, the elements of a geometrical construction can be expressed with the use of variables and dynamically manipulated by specially designed computational tools called variation tools.

In the research we were motivated to relate parts from different physical angle situations reminding the ones that an individual experiences in everyday circumstances in 3D space where such situations need not be distinguished. Our pedagogical was thus designed to provide opportunities for pupils to construct 3D geometrical figures and dynamically manipulate, transform and animate 3D objects often encountered in everyday situations (e.g. sliding doors) through Logo commands and variation tools. After a familiarisation phase with the basic Logo commands (Introductory phase), students were engaged in building rectangles using parametric procedures in at least two different planes of the Turtle Scene (Phase 1) and experimenting with variable procedures designed to create 3D simulations like doors, revolving doors and staircases (Phase 2).

The modalities of use of ETL's pedagogical plan offered a framework in which to account specifically:

- for meaning-making processes concerning angular relationships in the 3D space

- for student's learning trajectories and potential difficulties in coordinating different aspects of angle/turn concepts in 3D space.

Thus, it could be said that all the main educational goals envisaged a-priori have been achieved. In particular, the analysis of our data brought in the foreground the following three clusters of meanings constructed by pupils around the concept of angle.

Cluster 1: Angle as a slope while navigating the turtle in 3D space

The move of turtle in MaLT is interrelated with the conception of angle integrating two schemes based on turning:

(a) angle as a turn indicating both the act of body turning and the result of it, which inevitably involves directionality (dynamic scheme) and

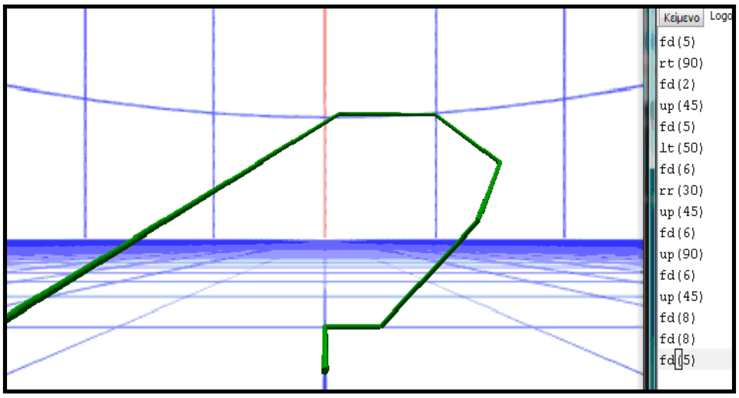
(b) angle as a turn represented by a number (measure scheme) (Clements et al., 1996).

During the introductory phase, students were asked to navigate the turtle in the 3D geometrical space of MaLT in such a way so as to simulate the take-off of an aircraft. In this particular task pupils focused on moving the turtle around and simultaneously appeared to connect this activity with everyday experiences and physical angle contexts. From the beginning pupils experimented with all the three sets of Logo turning commands[1] and it seemed that they had made links between the concept of angle as a turn with particular measure and that of angle as a slope.

The phrase *'Look there is a slope up(45) and then a slope of lt(50)'* in the following episode taken from the work of one group of pupils is indicative. As they didn't try to draw a geometrical figure but to navigate the turtle in a way so as to simulate the take off of an aeroplane, these pupils didn't focus on the internal angles of the crooked line that the turtle had drawn but at the angle that was drawn by the turtle in relation to the line of horizon. The significance of this constructive mathematical activity is set off if we take in mind that the standard angle concept first develops in situations where both arms of the angle are visible. Researches have also shown that only a small percentage of students can recognise the angle between the horizontal and a sloping surface where the one supporting edge is missing (Mitchelmore & White, 2000).

Episode 1

R: Hey, nice take off!! I see you hit the ground!



S1 Look there is a slope up(45) and then a slope of lt(50)

S2 Yes, that's true, slope 45 and slope 50. But also look when you change the length and not the slope the result is the same…the angle doesn't change.

They are working without speaking. They write

fd(5)

rt(90)

fd(2)

up(45)

fd(5)

lt(50)

fd(6)

rr(30)

up(45)

fd(6)

up(90)

fd(6)

up(45)

fd(8)

[1] In MaLT there are three kinds of turns: right/left turn in relation to turtle's trunk-vertical axis, rightroll/leftroll which moves the turtle around its trunk/vertical axis and uppitch/downpitch, which pitches the turtle's nose up and down.

fd(8)

fd(5)

However, a closer look to both students' dialogue and the Logo commands they used brings into the foreground their confusion in relation to the graphical results of the commands up(45) and lt(50). It seems that they focus in both cases to the angles drawn in relation to the line of horizon and not in relation to the previous position of the turtle, as it is the case. It could be pointed out that students oscillated between two different frames of reference:

- A world frame: defined in terms of directions 'up' and 'down'.
- A vehicle frame: typically associated with the orientation of a moving entity, here the turtle.

In the initial 'take-off' of the turtle the 'vehicle' frame of reference coincides with the 'world' frame of reference. In other words the 'up' in relation to the turtle's position coincides with the 'up' of the simulated 3D space. Then and especially after the command lr(30) the two frames contradict one another. That's why students kept on using 'up' command in order to get height but the result was the collision to the ground.

Based on indications like the above it follows that although 3D simulated space is closer to real life and every-day experiences, the body-syntonic metaphor appears to be less strong in 3D turtle geometry than in 2D. In other words when we move in real 3D space the up and down directions are usually stable (although not when we turn upside-down) because of gravity. Moreover, we walk in a 2D horizontal plane while the 3D turtle moves in different planes in 3D space. For instance, we can easily simulate 2D turtle motion with our body but we cannot simulate 3D turtle's motion. Thus, it seems that the body-syntonic frame which is inextricably linked with the world frame in real 3D space should be shrunk in favour of the 'vehicle frame' in the simulated movement of turtle in the 3D space.

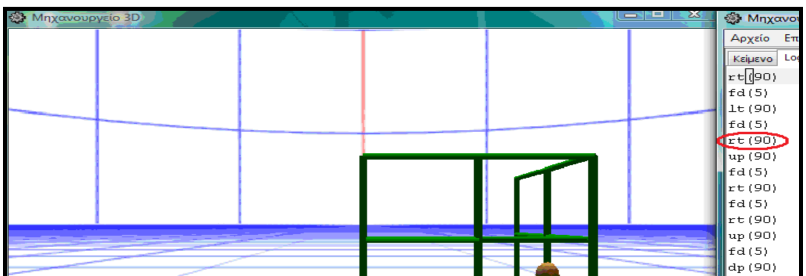
Cluster 2: Recognizing (or conceptualizing) a dihedral angle in 3D space

A second cluster of meanings concerned the conceptualisation of a dihedral angle in 3D space. This kind of activity appeared in Phase 1 of experimentation when the teacher/researcher asked pupils to construct rectangles using parametric procedures in at least two different planes of the Turtle Scene simulating the construction of windows in a virtual room. The need to design figures in different planes of the 3D space challenged pupils to move the focus of their attention from directed turns between lines and planes to directed turns between two similar geometrical figures.

The following episode is extracted from an activity where the students were trying to construct two windows in two consecutive walls (planes) of their own virtual room. The episode is indicative of the difficulties that students had faced at that phase while trying to approach a dihedral angle as a geometric shape in 3D space with a particular measure.

Episode 2

S1 I turned the turtle 90 degrees right in order to create an angle between the two walls. So, this is an angle between them.



S2 One angle…yes

R So, you turned the turtle in order to create…what exactly?

S2 It is like an open triangle…

S1 Hey…look! It is an acute angle! An acute angle!!!

S2 Yes, you are right. It is an acute angle because it is smaller than 90 degrees!

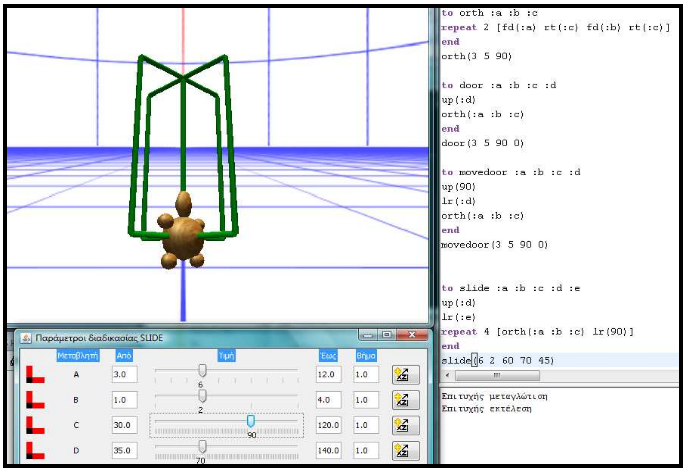
Students easily identified the dihedral angle defined by two consecutive windows (rectangles) and used the terminology familiar to them from 2D geometry lessons in order to describe it. However, they characterized the dihedral angle drawn by the turtle as an acute and not as a right one as it was the case, although they were they that had commanded the turtle to turn 90. It seems that they focused more on the visual characteristics of the figural representation and were confused by the ‘distortion’ of the dihedral angle as a result of the use of a vanishing point[2] in the line of horizon of the Turtle Scene designed to strengthen the sense of depth in the representation. Apart from the essential familiarisation with the new kinds of turtle turns (uppitch/downpitch, leftroll/rightroll) this interpretation could possibly be interpreted in the light of the fact that pupils who were accustomed to work with 2D representations of geometrical figures might have had difficulties in understanding the conventions used to represent a 3D object on the computer screen.

However, pupils seemed to overcome such misunderstandings through the dynamic manipulation of geometrical constructions which provided them with multiple perspectives of the same 3D geometrical object including dihedral angles created by 2D geometrical figures. The more the students appeared accustomed to the conventions used in the 3D simulated

[2] The main part of the ‘Turtle Scene’ component is a 3D grid-like interface. A perspective projection is used with three active vanishing points, so that a realistic effect of 3D space representation and navigation is created. The 3D space representation is based on the metaphor of hemisphere. The two active vanishing points have a deviation of 90^{o} and they are located in an iconic line of horizon which is conceived as the circumference of vertical circle. The third active vanishing point is conceived as the pole of the hemisphere.

space the more they were able to coordinate the visual characteristics of the dihedral angles with their measure related to the turtle's turns from one plane to another. The following episode is indicative:

Episode 3



S2: Yes, when we turn 90, an angle is formed (shows the angle between the two rectangles) 90.

S1: And from here (points at the variable c) this angle turns.

R: What is this 90?

S2: It is … a right angle!

R: Yes, but where is this angle?

S1: Between here…

R: So, where do you see this angle?

S1: Here that there are the 4.

S2: Among these …

R: So? Explain it to me clearly so that I can understand it.

S1: Among the 4 rectangles there are 4 angles.

In this case students were able to recognise the four consecutive right dihedral angles created between the four rectangles around x-axis. However, it should be stressed that MaLT functionalities and especially the simulation of the motion of rectangles (that represented a sliding door) around x-axis as a result of the use of 1d Variation Tool, gave students the chance to see the dihedral angles created from different perspectives. Viewing dihedral angles from different perspectives minimized the ‘distorting’ effects of the visual 3D representation and prompted students to focus more on the measure of the turtle’s turn in the Logo code so as to describe the angles as visualized in the composition of the four rectangles in 3D space.

Cluster 3: Angle as a dynamic entity for moving in different planes

A third cluster of meanings in our data analysis concerns the concept of angle as a dynamic entity for moving in different planes. Initially students have focused on changing planes as a result of changing turtle’s position. The use of the two new kinds of turtle turns (rightroll/leftroll, uppitch/downpitch) coupled with pupil’s experience in using variables and handling variation with 1d Variation Tool facilitated further the extension of their experimentation around the different positions of already designed 2d geometrical figures (e.g. a rectangle) in 3D space. This kind of activity appeared to provide a fruitful domain that challenged student’s intuitions and ideas about angle as a spatial quantity come into play since the use of these specific turns signalled a dynamic passage from one plane to another. For instance in the following episode, students initially decided to change turtle’s position and bring it up at a plane vertical to the horizontal before drawing the rectangle so as to simulate a door.

Episode 4

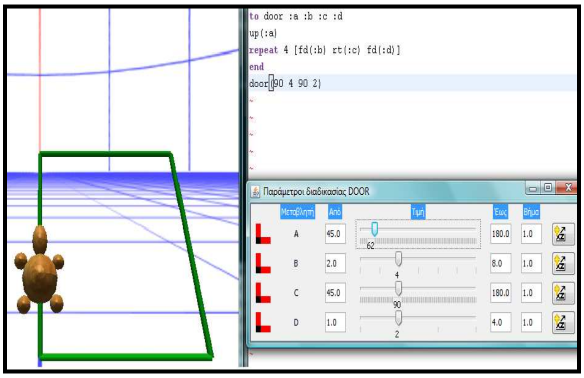
S1	For the door we will need a rectangle	
S2	Yes, we will need a rectangle … so as to make the rectangle. So, let’s try to make the door.	
S1	Fd;	
S2	No, firstly we will need a turn up…up (90). We will create a right angle.	He shows with his hands the change of planes.

Progressively the focus has been transferred to that of changing planes as a result of the change of position of a 2D geometrical figure in 3D space. At this point it is important to note that while changing planes only the final plane defined by the 2D figure is evident and as a result the initial position should be kept in mind in order to conceptualise angle’s measure (or vise versa). In many cases students decided to use not a fixed turn measure but a variable. Thus it seems that angle is approached no only as a dynamic turn in 3D space but also as a dynamic amount, in other words as a measure that can by dynamically handled and changed sequentially using the functionalities of 1d Variation Tool. In the following episode students decide to use a variable so as to progressively move the door that they have created in the horizontal plane to the vertical one.

Episode 5

S1	Lets do up
S2	a
S1	No, 90;

S2	No! a	
S1	Up [she moves her hand like moving a door]	
S2	Up…the whooole. So, what I need?	
S1	a	
S2	So, we will slowly create a door	She shows with her hand a progressive movement_of the rectangle between the horizontal and the vertical plane.
S1	up(:a) and now…	
S2	Now stop. We did up to create the angle, then forward, then right so now we need rt(:d) and then forward	to door a b c d up(:a) repeat 4 [fd(:b) rt(:c) fd(:b)] end door(90 4 90 2)



Students progressively got more and more capable of handling different aspects of angle simultaneously. For instance in the following episode students are experimenting with the variables of the procedure ‘Slide’ (which was given ready-made to them) so as to create a sliding door moving around. It seems that students create meanings in relation to angle:

- as a constitutive element of a figure which is defined and stay fixed (variable c)
- as a means to move from the horizontal plane to the vertical one in relation to the viewing axis of the user which is again defined and stay fixed (variable d)
- as a means of constantly changing levels around x axis (variable e)

SECTION B: Discussing the relation between students’ achievements and the use of the DDA in the context of the PP

The construction of mathematical meanings as a result of students’ interaction with a specific learning environment raises (among the others) semiotic issues related to the available representations. This perspective indicates that the relation which someone establishes between a representing and a corresponding represented is in its own right a process of meaning making. In other words it refers to a process of making sense of how a representing and a represented are related and if and how there is a link between a real object, its computational representation and the traditional means of representing mathematical objects in the classroom. Simultaneously students form gradually their own ‘understandings’ of the essence and the functionalities of the tool and develop schemes of use which were often quite different to those intended by the designer of the computational environment and the PP.

The analysis of our data has strengthened the above theoretical stance. Indicative of this is the episode one presented with more details in the previous sections. In short, it could be said that the description of the task at the introductory phase (the simulation of the take-off of an aircraft) had decisive implication on the ways by which students conceptualised angle as a slope while navigated the turtle in the 3D space. In addition, the world frame of reference which is inextricably linked to the body syntonic metaphor prevalent in 2D turtle geometry contradicted with the ‘vehicle’ frame of reference which is by design used in turtle’s navigation in the simulated 3D geometrical space of MaLT.

Other examples indicating the strong links between the available representations and functionalities in MaLT to the meanings generated by the students are the episodes 2 and 3. While students appeared to focus only on the visual cues of the Turtle Scene, they were confused about the measure of dihedral angles as a result of their distortion by the use of a vanishing point in the line of horizon providing a sense of dept on the screen. However the use of the 1d variation tool later on gave students the chance to view dihedral angles from different perspectives and to develop meanings concerning the mathematical structure of the visualised revolving door using respective arguments to describe the relation between the representing and the represented 3D object.

SECTION C: The relationship between what envisaged when planning the PP and the results of the TE.

Our main concern when planning the activities was to be open enough so as to leave space for the expression of students’ intuitions and for unrespectable answers to come about. Thus the multitude and divergence of the created meanings as described in the respective clusters (Part A) was something hoped for. However it could be said that there are two main points that were not acted out in the way we have envisaged:

- Time schedule. The introductory phase took more time than expected both because of contextual factors and because of the fact that we had underestimated the difficulties pupils might encounter when navigating an entity, the turtle, in a simulated 3D space through Logo commands. As a result the third activity of Phase 2 concerning the construction of a spiral staircase was not carried out.
- Student’s difficulties with specific representations. The main student’s difficulties that we observed seemed to be related to the conventions used in the simulation of the 3D space on the screen. For instance we remind the pupil’s difficulties to recognise the measure of a dihedral angle due to the visual characteristics of the respective representation (i.e. the existence of a vanishing point in the line of horizon (see episode 2).

A.4.6 Synthesis of ETL TE with MoPix

***<u>Section A:</u>* Students' achievements and the specification of the evidence supporting the claimed achievements**

One of the main educational goals presented in the MoPiX Teaching Experiment Guidelines concerned the students' construction of meanings about the role of the algebraic equations and the relationships between them in the context of changing a half-baked microworld. Students used MoPiX built-in and created MoPiX compatible equations so as to ascribe properties and behaviors to their objects and represent phenomena, such as collisions and motions.

On the basis of this educational goal, we classify the students' achievements into the following categories of analysis:

- Construction of meanings about the role of an equation through the interpretation of its symbols.
- Construction of meanings about the role of an equation through the editing of its symbols.
- Construction of meanings about the role of an equation through its conceptualization and development.

1. *<u>Construction of meanings about the role of an equation through the interpretation of its symbols</u>*

   At first place, students used the equations available in MoPiX without attributing any meaning to the symbols on the left or the right side of the equations. The criterion for selecting and using an equation was plainly its name. For example, students used the equation "amIHittingGround(ME,t)= (y(ME,t)≤(height(ME,t)÷2)) and Vy(ME,t)≠0" (faulty) presuming that it would make their object "hit the ground". Apart from the equation's name, all the symbols on the left and the right part of the equation were disregarded. In case that an equation's name consisted of symbols that didn't give a clear view regarding the behavior it would attribute to the objects (e.g "Ax"), the equation was disregarded as well.

   The next step in the construction of meanings about the role of the equations emerged when students started using equations after having attributed meaning *only* to certain of its symbols. In the case of the "Vx(ME,t)=Vx(ME,t-1)+Ax(ME,t)" equation, students didn't take into account the symbols on the right part of the equation. The decision to attribute it to their object was the result of a comparison between the left part of the equation at hand and the left part of the "Vx(ME,0)=3" equation. After attributing meaning to the symbol of "0" in the latter equation and using it to describe the object's initial velocity, the students sought for an equation to describe the object's velocity at any time. Since the left part of the "Vx(ME,t)=Vx(ME,t-1)+Ax(ME,t)" seemed to meet their needs, students decided to ascribe it to their object regardless of the meaning conveyed in the symbols on the right part of the equation.

   The use of an equation after having analyzed the meaning of each one of the symbols in an equation and explored the relationships between them was the last of the students' achievements in this category of analysis. In this case students viewed the equation as a set of symbols that combined into unified whole. The kind of behavior an equation would give to an object was determined after having attributed meaning to each one of the symbols and having defined the symbol's specific role in the equation.

2. *Construction of meanings about the role of an equation through the editing of its symbols.*

The second category of achievements refers to the construction of meanings about the role of an equation through the editing of its content. By "editing the content of an equation", we mean the process in which students performed changes to the symbols composing an already existing equation but left the structure of the original equation intact.

Students edited the already existing equations for two distinct reasons: so as to attribute meaning to certain symbols of the equation after comparing the effect that the new equation had on objects with the effect of the original one and -after having attributed meaning to all of the equation symbols- so as to express their ideas and generate behaviors for their objects that were not accurately described by any of the already existing equations.

The elements that the students often altered in an equation were the arithmetic values present on its left or right part. The arithmetic value editing they performed could be classified into two categories: editing so as to replace the existing arithmetic value with a different one and editing so as to replace the arithmetic value with a variable.

The students of the 3rd workgroup, after using the MoPiX Library equations to define their object's motion in the horizontal axis, they sought for equations that would make their objects move in the vertical axis. The first equation they detected at the Library was the "Vy(ME,0)=0", an equation that describes the initial vertical velocity of the object. After attributing the equation to the object and watching the animation generated, students decided that the equation they had chosen wouldn't move their object for two reasons. The first one concerned the arithmetic value on the right part of the equation. The "0" had to change into "3", so as for the object to have a velocity in the Y axis. The second one became apparent after attributing the "Vy(ME,0)=3" equation to the object and concerned the arithmetic value on the left part of the equation. The "0" value on the left part that referred to the time instance had to change and so as for the object's velocity to be "3" at the following time instances as well. As students looked for ways to incorporate the "all the next time instances to come" element in their equation, they decided that they needed a symbol which they would "just look at and understand that it represents the infinity". The equation they formed was the "Vy(ME,t)=3".

3. *Construction of meanings about the role of an equation through its conceptualization and development.*

The third category of achievements refers to the construction of meanings about the role of an equation through its conceptualization and development. The difference between this category and the previous one lies in the fact that, in this case, students didn't just change an already existing equation but actually constructed an equation from scratch, using the MoPiX mathematical formalism. This means that in order to express their ideas about the behavior they would like to give their objects, students invented new symbols to which they attributed meaning and related these new symbols to already existing ones, forming a completely new equation. As it becomes apparent, in this case, students not only decided on the content of an equation, but also decided on its structure.

The students of the 1st workgroup decided that they would like to link two of their microworld's objects and make them interact under certain circumstances. The idea was to create two equations that would oblige one of the objects to respond to specific events handled by the user. The students decided both on the event that would force the object to respond (i.e the change in another object's position) and on the kind of the reaction such an event would cause (i.e changes in the object's colour). In this process students not only determined the content of the equation (the kind of symbols they would include) but also defined the equation's structure (the ways in which the symbols would be related to each other). Moreover, since no other symbol could describe the effect they would like to generate, students had to invent new symbols to which they attributed meaning, defined the values they would accept and used them so as to relate the new equations to each other.

<u>Section B:</u> ***The relation between the students' achievements and the use of the DDA in the context of the PP***

As it was mentioned in the Teaching Experiment Guidelines, we chose to replace the term "representations" that appeared in the Common Research Question with the phase "(students)…using the representations", in order to depict the importance of the use of the DDA in the students' construction of meanings. This specific choice was also supported by the fact that the activities we had designed for our PP and the microworld that we had developed (i.e a half-baked microworld), by its own nature, called for the use of the DDA's representations as it provided students deep structural access to its functionalities. Thus, we expected the students' achievements to be inextricably interwoven to the use the representations available in MoPiX.

Specifically, in order to construct meanings about the role of an equation, students used the DDA's symbolic representation system (i.e MoPiX equations) in the process of:

- Interpreting the role of certain symbols in an equation or interpreting the equation itself as a unified whole,
- Editing the symbols of an already existing equation (modifying the arithmetic values present in the equation and replacing them with another arithmetic value or a variable),
- Constructing a new equation (conceptualizing and developing an equation from scratch, deciding on its structure and content).

In each one of the processes described above, students, apart from using the symbolic representation system, also used the graphical one. The graphical representation generated by the execution of the equations attributed to the objects was not used so as to directly express ideas as it was the fact for the symbolic representation system, but it was used so as to:

- Attribute meaning to an equation -or certain of its symbols- after adding it or removing it from an object,
- Verify the role of an already existing equation or the role of newly formed one,
- Decide on further changes on a newly formed equation regarding its structure or content.

In any case, the two MoPiX representation systems were used interchangeably by the students in the process of changing a half-baked microworld and both contributed to the student's construction of meanings about the role of the equations.

<u>Section C:</u> ***Relationship between what we envisaged when planning the PP and the actual results of the TE***

One of the main choices we made during the development of the Pedagogical Plan was not to give out a certain set of activities that students would have to accomplish before moving on to the next one. On the contrary, drawing on the constructionist viewpoint, we decided that we wanted the experimentation process to take place in a context that would allow students to construct meanings about the role of the equations by themselves and at the same time permit them to engage in activities that would be personally meaningful to them and not imposed upon them. The half-baked microworld we developed for the implementation of our Pedagogical Plan supported our choice as it called for changes that could result in the construction of a different and unique artifact for each of the workgroups.

Taking into consideration this perspective, we expected students' achievements to differ for each workgroup according to the trajectories they would choose to follow. For example some of the teams didn't attribute meaning solely to certain symbols of an equation before viewing it a set of symbols related to each other. Others attempted to edit an equation even before attributing meaning to its symbols. Moreover, the students of one of the workgroups selected not to change the microworld in terms of constructing new equations. They created numerous new objects to which they ascribed almost every equation already existing in the Equations Library so as to make their microworld look like a "fun fair".

As it becomes apparent, we were not astonished by the fact that the students constructed meanings about the role of the equation in ways different to each other's. By adopting the constructivist framework and the using theoretical construct of the half-baked microworld (i.e microworlds especially designed for instrumentalization) for the implementation of the PP, we actually pursued that kind of diversity to emerge. However, we were surprised to find out that students constructed meanings about the role of the equations in ways that we hadn't initially thought of.

A.4.7 Synthesis of IoE TE with Mopix

1. Student achievements

The envisaged educational goal of the teaching experiment was the development of students' concepts of motion in accordance with Newtonian laws. In the implemented pedagogical plan, this focused primarily on the development of concepts of velocity and acceleration. Specifically:

- velocity as change in displacement
- velocity (in a plane) as a two dimensional vector, either (magnitude, direction) or (horizontal magnitude, vertical magnitude) - the second of these being most naturally encoded in MoPiX notation
- velocity remains constant unless acted upon
- acceleration as change in velocity
- acceleration as a force - specifically acceleration applied at an instant

Through the course of the experiment, students' ways of talking and writing about velocity and acceleration changed in ways consistent with this educational goal, though their use of acceleration was much less secure. Their use of MoPiX showed that they were able to operate with these concepts in order to build models that moved in ways compatible with their intentions, though the nature of this varied between students and achievement was uneven. We would not claim that all students achieved to the same extent. The types of achievements we consider relevant include:

a) Separate treatment of horizontal and vertical components of velocity and acceleration in order to describe motion.

By later sessions, students' problem solving processes while using MoPiX consistently dealt separately with vertical and horizontal components of motion when adding and editing equations to models. Moreover, when using other modes of communication, students also described motion in terms of x and y components, making use of the terms Vx (or 'x velocity') and Vy and, to a lesser extent, Ax and Ay. As may be seen in example 1 below, this allowed descriptions of motion that were more analytical and consistent with the principles identified above.

Example 1 The following task was given both in the written pre-questionnaire and in the post-questionnaire:

> Imagine throwing a tennis ball against a wall. Describe n words how the ball moves and how its motion changes.

Art responded to the pre-questionnaire task:

> The ball flies towards the wall losing height then it hits the wall losing some energy to the wall out as sound, bounces off the wall continues falling but in a different direction.

and to the post-questionnaire task:

> As it is flying towards the wall its x velocity doesn't change while the y velocity is decreasing. When the ball hits the wall the x velocity changes direction (becomes negative) and some energy is lost to the wall, the y velocity keeps decreasing at the rate of -9.8. As the ball hits the ground y velocity changes direction

Art's responses before and after the teaching experiment show some similarity in the use of the idea of 'flying' towards the wall and losing energy to the wall (a concept presumably drawn from his lessons in Physics as his use of MoPiX had not included this phenomenon). However, his response to the post-questionnaire (i) presents velocity as a vector quantity, separated into horizontal and vertical components (ii) recognises that the horizontal velocity does not change until it hits the wall (iii) identifies bouncing off a vertical or horizontal surface as a change of sign of the horizontal or vertical velocity respectively (iv) recognises that the vertical velocity is affected by the constant acceleration of gravity.

c) Development and use of the concept of acceleration is more fragile than that of velocity.

Students quickly developed systematic strategies to construct models involving only velocity, analysing the values needed to produce the desired effects. In general, they struggled to solve problems involving acceleration and were inconsistent in the ways in which they talked about it and applied it. This may have been at least in part because acceleration was addressed later in the teaching sequence. Example 2 illustrates the difference.

Example 2. During Session 5 students were able to use changes in velocity in order to change the direction of motion of objects. In Session 7, they were asked to achieve changes in direction by applying an acceleration at an instant. Aa chose first to work on the problem of drawing a square using changes in velocity in order to change direction, he then revisited the same task of drawing a square by using acceleration as a force applied at an instant in order to achieve the same effect. In each case, Aa started by using a trial and improvement approach in order to make the first corner but then used systematic methods to turn subsequent corners. When using velocity, his progress through the trial and improvement stage was rapid, using systematic methods to correct errors. The only errors made on turning subsequent corners were errors of sign and by the final corners he was changing both horizontal and vertical components of the velocity without making intermediate trials. When using acceleration, the initial trial and improvement stage was much longer, involving a high number of trials, some of which did not appear systematic. Having achieved the first turn, his methods appeared more systematic but much slower than when using velocity directly. Towards the end of this task, he spent several minutes carefully examining the set of equations, pointing repeatedly to the velocity equations as if recalculating the horizontal and vertical velocities at each application of an acceleration.

c) Operationalisation of the concept of acceleration as change in velocity appears to be supported by some forms of semiotic resources more than by others.

Students' ways of talking about velocity and acceleration and their use of these in problem solving varied across the course of the teaching experiment and across the various modes of communication in use. This aspect is still subject to fuller analysis but we present example 3 here to illustrate the way in which different modes of communication may affect the meanings constructed for acceleration.

Example 3. While working on question 3 of the post-questionnaire (see below), Ab and Aa made use of the diagram provided, interacting with it with speech and gestures. They also made use of a calculator, pencil and paper and MoPiX. When using the diagram, they struggled with the idea of constant acceleration, which seemed to conflict with their interpretation of the diagram. Ab seems to confuse acceleration with velocity:

> it's decelerating here [slides from t=50 to t=130 LH] then here it's zero here [points LH and RH at t=130 (prolonged)] and starts accelerating again [rapid slide from t=130 to t=150 RH]

The diagram and interaction with the diagram using gesture to mimic the imagined motion of the ball provided resources that did not enable the students to distinguish clearly between acceleration and velocity. They did not distinguish between horizontal and vertical components and associated upward movement with acceleration, even moving the sliding finger faster as it moved upwards.

As they started to fill in the table, however, renewed interaction with the wording of the question led them to fill the Ax column with zeros and the Ay column with -0.1 all the way down. Re-visiting the wording of the question prompted the students to separate acceleration in the horizontal and the vertical directions and to operate with them as constants. The use of the verbal and symbolic modes rather than the diagrammatic enabled them to complete the acceleration values in the table correctly, apparently in contradiction to their earlier ideas.

After an initial attempt to complete the Vy column by considering the diagram, they decided to calculate instead. Aa got out his calculator and prepared to do some calculations. With the calculator by his side, he developed the approach he intended to take, communicating with his partner in interaction with both table and diagram:

> if you got y acceleration at -0.1 here [points to Ay at t=0 in the table] to find out at what point it stops here [points to t=50 on diagram] if you times that [points to Ay at t=0 in the table (-0.1)] by the time taken to reach here [points to t=50 on diagram] .. you should get the velocity for the y

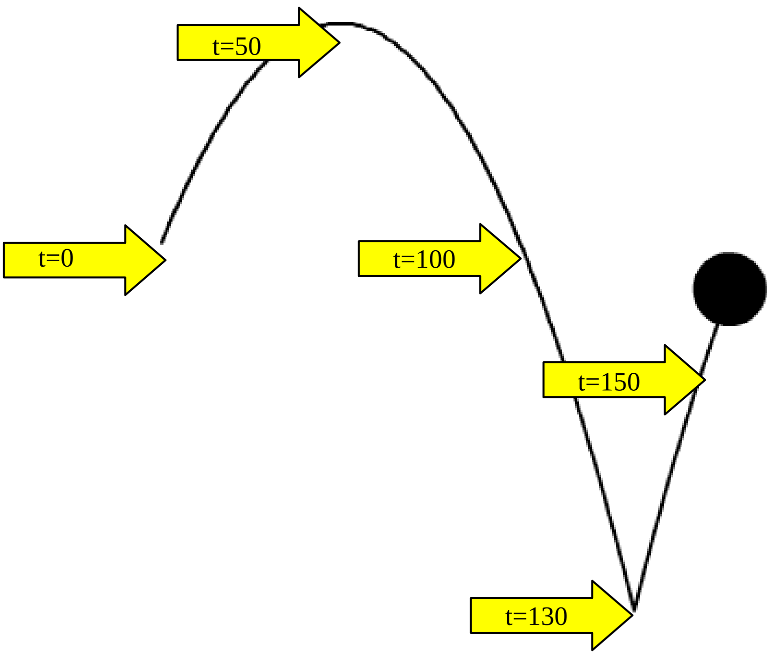
Having decided to calculate, the affordances of the calculator itself allowed connections to be made between, on the one hand, the symbolic mode of the table and, on the other hand, the diagram. Pointing at the diagram now served to identify a point in time, rather than a movement.

This episode illustrates the fragility of the notion of acceleration for these students. It was only with the support of a range of interacting semiotic resources that they could be successful in resolving the problem.

3. The diagram below shows the path of a ball thrown into the air and then bouncing off the ground.

The ball's initial velocity (at time t=0) is 2 in the x-direction and 5 in the y-direction. Its acceleration is -0.1 in the y-direction (a MoPiX approximation for gravity).

Complete the table below with the velocity and acceleration of the ball at the given times.



time	velocity		acceleration	
	Vx	Vy	Ax	Ay
t=0	2	5	0	-0.1
t=50				
t=100				
t=130				
t=150				

2. Relationship between students' achievements and the use of the DDA

According to our social semiotic theoretical framework, different semiotic systems provide different meaning potentials. The MoPiX 'languages' of equations and animations emphasise specific meanings related to motion, as indicated below, that are not supported so strongly by everyday forms of language. The lexico-grammar of this new language enables particular kinds of statements about velocity and acceleration. A major change in students' ways of talking about velocity and acceleration was their adoption of the MoPiX formalism in their talk as well as in actually programming. This contributed to their changing ways of communicating about motion, in particular the higher degree of analysis of components of motion. Characteristics of the DDA that we see reflected in the students' achievements include:

a) Separate equations and notation for horizontal and vertical components of position, velocity and acceleration. This provides students with a means of talking about velocity as a vector, moving them away from the 'everyday' language ways of talking about velocity that make it difficult to distinguish clearly from speed.

b) In order to build an object that moves it must have a velocity assigned to it. This makes velocity an explicit part of the description of motion (as seen, for example, in example 1 above).

c) Unless the velocity is changed, the object will continue moving in a straight line. Velocity may be changed by explicit statement of a new velocity (at a specific time), by applying an acceleration (either a constant acceleration or at an instant), or by applying an equation defining a change in velocity resulting from interaction with another object (e.g. changing the sign of velocity in order to 'bounce' off a wall). A self-imposed challenge for one pair of students during session 2 was to make a moving object stop. Their limited experience with MoPiX even at that stage enabled them to identify the problem as constructing an equation to apply a new velocity at a specific time. Although they did not have sufficient grasp of the syntax to complete the task during that session, the structures provided by the language of equations enabled them to analyse the situation.

d) The quantification of velocity and acceleration also allows problem solving strategies that involve calculation. In some cases this was achieved through examination of sets of equations within MoPiX (as we may suppose happened with Aa at the end of the episode described in example 2). One student appeared to use this strategy regularly, examining sets of equations extensively before running them. However, he was unusual and other students tended to use more trial and improvement before attempting to analyse quantitatively. In other cases, quantitative analysis seemed to need to be supported by other semiotic means, including pencil and paper or calculator (as in example 3).

3. Relationship between what was envisaged when planning the PP and the actual results of the Teaching Experiment

The development of use of velocity and acceleration to build animations worked much as anticipated and students clearly made use of the MoPiX formalism to communicate about velocity and acceleration.

In the original PP we anticipated more attention would be given to interaction between objects. In practice, the complexity of the equations needed to achieve the kinds of interactions the students were interested in was too great to be handled easily within the editor and gave rise to much frustration. In later sessions, we thus avoided all but the simplest interactions.

In designing the PP, we intended to give time to the students to work in groups on more substantial projects with self-determined goals. A combination of factors prevented this happening:

- tasks that were intended to be introductory proved to be more challenging than anticipated and became a substantial focus of attention for students and for us
- continuity between sessions was hard to sustain as some students only attended intermittently
- students were unused to working in groups and there was not enough time to establish effective patterns of working that would have supported projects
- the extra-curricular organisation of the sessions and the pressure of examinations and attendance at interviews for university entrance meant that, although students appeared highly motivated, they were unable to sustain work on projects outside the scheduled sessions.

A.4.8 Synthesis of IoE TE with MaLT

1. Students achievements

The envisaged educational goal of the teaching experiment was to investigate the meanings students make in relation to the three dimensional geometry through their semiotic activity or, in other words, we are exploring students' interaction with 3D geometrical shapes within MaLT and other modes. In particular we are interested in

- The different modes of communication students make use of when interacting with the 3D geometrical shapes,
- The choices students make between and within semiotic systems (modes) in order to communicate their completed design to their peers.
- The ways in which the properties of shapes are represented in different semiotic systems.

Our collected data was constrained in many factors, including: the time available, the curriculum constraints and the students' level of achievement. This affects our ability to use the data to address the research questions. Moreover we still in the process of analysing the data. We have started by selecting an episode from the implementation stage where students had already made use of MaLT in previous sessions. In this episode students are trying to design doors of their project which is a sport centre. The student we video recorded is trying to design a revolving door using MaLT. The session was conducted in the computer (ICT) room in the school. The student has a help sheet to guide her on how to construct the door.

In addition to that we collected students' paper work through the implementation of MaLT in the school and the posters they produced as their project and we video recorded their presentations of these posters to the whole class as well.

We designed the educational plan in coordination with the teacher where she started pre-experiment activities with the students such as the doing some measurements and looking at plans and elevations of their school and they agreed to design a sport centre. Afterwards, the students had been introduced to different drawings (plans and elevations) to be familiar with these notions. Later, a hands-on session was held where the students used multilink models to represent some plans and elevation. At that stage of work students 'enjoyed' the tasks and they engaged in the tasks. However, we did not succeed to encourage them to work as groups especially the selected group for data collection. But students, in general, used the plans and elevations terms successfully either in their drawings or in their interaction with each others and with the teacher/researchers.

Working with MaLT started after that stage. Our analysis is still at a very early stage, so most of our comments here are only preliminary. We need to deploy all of the collected data to answer the question of students' achievement. However, we can point to two main achievements:

a) The ability to use the terms plans and elevations and to draw them to different types of objects.

Students were able to talk about different representations of buildings, identifying different representations of the same building and justifying their identifications.

They were successful in drawing plans and elevations of multilink models from different points of view; constructing models from plans and elevations (from different points of view as well); matching architects' plans to pictures/photographs of buildings; and drawing upstairs and downstairs floor plans of their own designs, though these did not always match other features of their designs.

b) Designing their posters and communicating these posters with others.

In the last session students worked in groups to design their posters in order to present them for other groups. The ways in which they designed their posters reflected the multisemiotic nature of the activities the students went through this experiment. Most of the posters include writing, images using several forms of representation, MaLT procedures, print outs of a MaLT image showing the door.

2. Relationship between students' achievements and the use of the DDA

We are not able to answer this question at this stage of the analysis.

3. Relationship between what was envisaged when planning the PP and the actual results of the Teaching Experiment

Although the PP was planned in conjunction with the teacher, it was overambitious in terms of what it was possible to achieve. In particular, the time available was insufficient and the students had low confidence and, in many cases, poor motivation, and were unused to collaborating with one another. Their lack of previous experience with Logo meant that progress with MaLT was slower than anticipated. Moreover, because of students' lack of independence and poor social skills, the quality of the data does not provide as much access to their choices and communication strategies as we originally envisaged.

A.4.9 Synthesis of ITD TE with Alnuset

Students' achievements

1a. achievements with respect to the a-priori envisaged ed. goals.

The goals considered in the a-priori analysis are related to the development of the capability to practice a semantic control over algebraic expressions and propositions solving problems of algebraic nature.

Let us consider some of these didactical goals listed below:

- Learning to practice the control of what variables and algebraic expression indicate in an indeterminate way within a numeric domain
- Learning to practice the control of the relationship between two expressions using quantitative and formal methods for distinguishing among equivalent expressions, opposite expression and reciprocal expressions
- Constructing a meaning for the notion of roots of polynomial and understanding the link between the roots and its factorisation
- Constructing a meaning for the notions of conditioned equality, equation, equivalent equations, truth value of equation, truth set of an equation

The TE has demonstrated that it is possible to achieve these didactical goals exploiting the mediation of Alnuset.

In the context of our work the educational goals are considered as achieved when students show to be able:

- o to use the ways in which the expressions and propositions are represented in Alnuset to solve the task proposed, showing to control them on a operative and semantic level.
- o to justify the contradiction emerged in the activity making reference to the representative events mediated by Alnuset
- o to use correctly the terminology introduced by the teacher to indicate specific algebraic notions both on the protocols and in the dialogue with the other participant in the teaching learning activity

1b. specification of the evidence supporting the claimed achievements

This is an example that gives evidence of how the use of the characteristics of Alnuset has mediated the achievement of some envisaged educational goal

This is one of the first task assigned to the students

> *Consider the following assertion: The two expressions* $-\mathbf{x}$ *and* $-\mathbf{x}^2$ *considered in the rational numbers set always represent a negative number. What do you think about this statement? Justify your answer.*
>
> *Construct the two expressions on the AL and verify your answer. Then try to justify it using what is displayed on the AL during the interaction. Is there any difference among the following:*
>
> $-x^2$ *and* $(-x)^2$ *and* $-(x)^2$*? Use the bi-dimensional editor of ALNUSET to represent these expressions on the AL and verify your answer.*

The educational goal of this task is the development of the control of what an expression indicates understanding that what it indicates is determined from the sequence of the operations that are present in its form.

The majority of students answer that $-x^2$ is a number always positive because the even power of a negative number is positive. In this answer there are two errors: the first one is that $–x$ is considered as a negative number, the second one is that the power is interpreted as it was $(-x)^2$.

They represent the expression on the algebraic line, drag the variable x and observe that the point corresponding to $-x^2$ on the algebraic line is always located on negative numbers.

Some their comments are reported:

> *We have verified with Alnuset that what we have written is false, so the assertion reported in the text that $-x^2$ is always negative is true.*
>
> *With Alnuset we have verified that $-x^2$ is a negative number, $(-x)^2$ is a positive number and $-(-x)^2$ is a negative number coincident with $-x^2$*
>
> Some student are quite amazed of these results
>
> A pair of students write: *$-x^2$ and $(-x)^2$are the same thing because making the square you obtain always a positive number…* and after the verification with Alnuset …*Ah, hence they are not the same thing, because in one expression the minus sign is inside the parenthesis while in the other no.*

Through the use of Alnuset contradictions emerge in the Activity.

Students try to overcome them interpreting differently the algebraic expressions and comparing their interpretation with those of other students.

The achievement of the educational goal is the result of a double level of mediation of Alnuset, namely the mediation to the student's action and the mediation to the communication among students and teachers

2. section discussing the relation between students' achievements and the use of the DDA in the context of the PP. In this section the issue of 'representation' should be addressed, according the different theoretical approaches that each team adopt
To discuss the relation between students' achievements and the use of the DDA in the context of the PP let us consider some didactical goals.

Some didactical goals of our PP regard the semantic control over expressions with the aim to recognize when they are equivalent, opposite or reciprocal and to demonstrate these their relationships.

The quantitative approach mediated by the algebraic line allows the student to discover that two expressions are:

- equivalent when:
    - They make reference to the same point on the line and they belong to the same post-it of the point when the variable from which they depend on is dragged on the line
- opposite when:

- o Their respective points on the line are always symmetric each others with respect the point 0, when the variable from which they depend on is dragged on the line
- o The point on the line corresponding the sum of the two expression is always 0, when the variable from which they depend on is dragged on the line

- reciprocal when:
  - o The point on the line corresponding the product of the two expression is always 1, when the variable from which they depend on is dragged on the line

Through the algebra of formal operation mediated by the AM, students can experience that two expression A and B are:

- o equivalent when it is possible to demonstrate that they have a common form, namely that A=B
- o opposite when it is possible to demonstrate that A+B is equivalent to 0 or when A=-B
- o reciprocal when it is possible to demonstrate that A*B is equivalent to 1 or A=1/B

Using the AM of Alnuset it is possible to perform these three demonstrations using rule of transformation that make reference to the properties of basis of the operation and in particular the property:

- o A+-A=0 to demonstrate the opposite relationship between expressions
- o A*1/A=1 to demonstrate the reciprocal relationship between expressions

In order to analyze the relationship between the characteristics of the DDA and the students' achievements we look for evidence on how the operative functions and the representative events have mediated

- o the arising of objective for the tasks at hand both on the quantitative and a formal level
- o the capability to justify the solution performed using the representative event mediated by Alnuset to refer to algebraic notions , meanings and referential objects involved in the task
- o The capability to use correctly the terminology introduced by the teachers during the development of the activity.

3. **section addressing the issue of the relationship between what envisaged when planning the PPs and the actual results of the TE:**

A clear and complete analysis of this relationship has not yet been elaborated.

On the basis of a first analysis of the data we can state that the students' achievement reflects what we have envisaged in terms of didactical goals when planning our PP.

As far as the role of mediation attributed to the two environment of Alnuset in the design of the PP, the results of our observation confirms that role. The TE has evidenced that:

- The Algebraic Line is really useful to develop the capability to practice a semantic control over variables, expressions and propositions, namely the control of their relationships with their referential objects from a quantitative point of view
- The Algebraic Manipulator is really useful to develop the capability to operate with expressions and proposition maintaining a semantic control over the actions performed on them
- These two environment can be fruitful used to integrate the quantitative approach and a formal approach in the didactical practices of Algebra
- In the AM there was an aspect of its functioning that could cause didactical obstacles and that we have modified (see successive section).

As far as the design of the sequence of tasks to be assigned to the students is concenrned, the development of the TE has convinced us to work out some minor modification on the cards for the students containing the text of the tasks(i.e. change in some requests) and to change in some cases the time dedicated to the solutions of some task.

Finally we observe that the development of the TE has allowed us to refining our theoretical assumption to justify and to put into context the learning phenomena emerging with the mediation of Alnuset. At the moment this elaboration is in progress.

Some of the actual findings of the TE were not envisaged a priori

Through the TE we want to evaluated not only the students' achievements but also the educational effectiveness of the artefact to highlight aspects of its functioning that create obstacles on the didactical level with the aim to modify it.

In the following we present an episode that occurs in the TE that has obliged us to modify an important aspect of the AM functioning.

This is a task proposed to the students in our experimentation.

Represent these two expression on the algebraic line:

(x^2+x-2)/(x+2);   x-1

Use the drag function and the tracking function to verify the relationship between these two expression

Write what you ought to do to formally highlight their relationship

Use the algebraic line and the symbolic manipulator of Alnuset to highlight their relationship

The experimentation has shown that the algebraic line is very useful to explore and discover the relationship of equivalence with restriction between two expressions.

It is important to note that the two expressions make reference to the same point on the line and are contained onto the same post-it for any value of x except that for x=-2. When the variable x is dragged on the point-2 the first expression disappears from the post-it and from the line. This representative event is source of demands, interpretations and justification

Moreover, to formally demonstrate their equivalence it is necessary to find the roots of the polynomial x^2+x-2 on the algebraic line by means of a specific function and then to transform (x^2+x-2)/(x+2) into x-1, first factorizing x^2+x-2on the basis of its roots previously determined and then simplifying the expression.

The experimentation has shown a limit in the functioning of the software that can create obstacles in the management of the didactical situation. This extract of dialogue between the teacher and a student witnesses this limit.

Student: excuse me , if the two expressions are equivalent why x-1 exists on the algebraic line when x=-2. Also x-1 is ought to disappear?

The student refers to the following transformation she realized:

$$\frac{x^2+x-2}{x+2}$$

$$\frac{(x+2)\cdot(x-1)}{x+2}$$

$$\frac{(x-1)\cdot(x+2)}{x+2}$$

$$(x-1)\cdot(x+2)\cdot\frac{1}{x+2}$$

$$(x-1)\cdot 1$$

$$(x-1)$$

Student: if I simplify this expression the result is x-1. Hence, they are equivalent , so also in this expression ought to be a restriction.

Teacher comment : I have to admit that I was in difficulty to answer.

This episode together with other ones have brought the team to modify the functioning of the symbolic manipulator.

Before the experimentation, inserting the expression *(x^2+x-2)/(x+2*) into the symbolic manipulator environment the interpretation of the restriction was under the responsibility of the user.

Now inserting the expression *(x^2+x-2)/(x+2*) into the symbolic manipulator environment the computer automatically pose the condition x+2≠0 and the algebraic transformation can be realized in the following way:

$$\begin{cases}\dfrac{x^2+x-2}{x+2} & if \;\; x+2 \neq 0\end{cases}$$

$$\begin{cases}\dfrac{(x+2)\cdot(x-1)}{x+2} & if \;\; x+2 \neq 0\end{cases}$$

$$\begin{cases}\dfrac{(x-1)\cdot(x+2)}{x+2} & if \;\; x+2 \neq 0\end{cases}$$

$$\begin{cases}(x-1)\cdot(x+2)\cdot\dfrac{1}{x+2} & if \;\; x+2 \neq 0\end{cases}$$

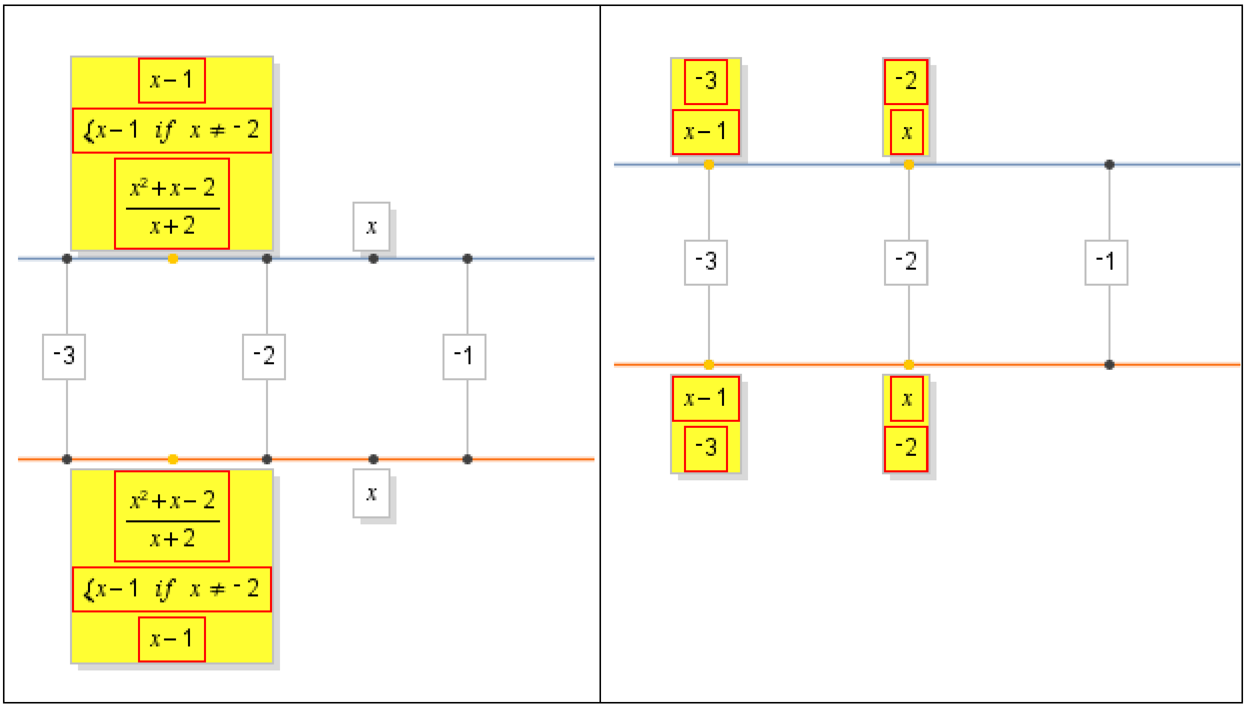
$$\{(x-1)\cdot 1 \;\; if \;\; x+2 \neq 0$$

$$\{(x-1) \;\; if \;\; x+2 \neq 0$$

$$\{x-1 \;\; if \;\; x+2 \neq 0$$

$$\{x-1 \;\; if \;\; x \neq -2$$

Finally, we observe that the two expressions x-1 e x-1 if x≠-2,whether represented on the algebraic line have different behaviours as shown from the following images that highlight the usefulness in the context of the problem assigned to the students.



A.4.10 Synthesis of ITD TE with Aplusix

§1. Educational goals and students' achievements

The educational goal envisaged in the design of our PP was for students to understand the structure of numerical expressions and in particular to:

- Learn how to represent a numerical expression as a tree;
- Learn how to “build” a tree given a numerical expression or an expression described in natural language;
- Learn how to “read” an expression represented by a tree;
- Learn that there is only one linear expression for a tree, while there may be different tree representations of an expression represented in linear form.

At the basis of our PP design was the hypothesis that the tree representations mediated by Aplusix may be useful for reaching these educational goals.

§1a. Achievements with respect to the ed. goals envisaged a-priori

Analysis of the TE and comparison of initial and final test results have revealed some development in students’ procedural competencies regarding numerical expressions with respect to the initial test results. At the end of the experiment some students were able to manage the “systemic” structure of expressions: they could identify the main operation of an expression, they could understand the priority of some operations with respect to others, and could sometimes identify equivalent expressions.

With respect to the educational aim the following results were observed in the final test analysis:

- The majority of students were able to build tree representations for numerical expressions. Note that at the beginning of the experiment students were unfamiliar with tree representations of linear expressions.
- Most students learned how to “read” an expression represented by a tree, at least for simple expressions. Nevertheless, some difficulties concerning the use of parentheses for expression translation are still evident in students’ behaviour: some students are unable to control parentheses even if they are able to do calculus correctly (See test results)
- The language used by students to describe expressions is clearer and more appropriate with respect to the initial test. In the initial test, students translate the expression respecting the “stenographic” way of the expression at hand, while in the final test some students are able to use the ideographic nature of algebraic language. For example, in the final test, Giulia described the expression 2*(3-1)+4/2 by writing “two times the difference between 3 and 1 is added to the quotient between 4 and 2”. In the initial test she described the expression (5-1)/4 writing “I subtract 5-1 and I divide the result by 4”.
- Some students learned to identify equivalent expressions from a structural point of view by comparing tree representations and linear representation of an expression.

§1b. Specification of the evidence supporting the claimed achievements

In general, student achievement is revealed by analysis of student protocols and comparison of initial and final test results. In the following we report the analysis of test results.

Test results

Student pair work

Two tasks (Task 1 and Task 2) were proposed in the initial test and in the final test concerning the relationship between linear expressions and expressions in natural language. In the final test a further task (Task 3) was proposed concerning tree representation of expressions. Our analysis is based on comparison of students' results from the corresponding tasks in the two tests.

For each task (Task 1, Task 2, Task 3) we present:

- A description of the task
- A table containing students' results in solving the task in the initial test
- A table containing students results in solving the task in the final test.

Task 1

Students are divided into pairs. One in each pair is a given two linear expressions like (2*3)+(5*3) to write in natural language and is then to read the text to their partner, who translates the statements into linear expressions. At the end, each pair compares the two linear expressions (the translated expressions and the given expressions) to see if they coincide. The task is then repeated with roles reversed.

Initial test

Answers	Pair of students	Observations
Correct answers	4	Stenographic translation ***
Incorrect answers	3 (more than 1 exercise out of 4)	Parentheses were not present in the translated expressions, fraction symbols were not positioned in the correct place
Invalid answer	1	Teacher helped students to solve the test

***This translation is performed by stenographic writing, disregarding the structure of the expression. For example, instead of translating the expression (2*3)+(5*3) as "the addition of 2 times 3 and 5 times 3", students wrote something like "you have to open the brackets, then you write 2 times 3, you close the brackets and add another bracket with 5 times 3.

Final test

In the final test the expression that students had to translate into natural language were more difficult that those proposed in the initial test (for example 2*(3+5)-4*(14-10) or $\frac{1+3(5-1)}{4-(3-1)}$)

Answers	Pairs of students	Observations
Correct answers	3	
Incorrect answers	3 (1 exercise out of 4) 1 (3 exercises out of 4)	Errors mainly concern the use of parentheses and the priority of operations***
Not valuable answer	1	

***Some mistakes concern the translation of parentheses. To translate the expression 2*(3+5)-4*(14-10) at least two students wrote “you can do 2 times 3 plus 5, then you can subtract 4 times the difference between 14 and 10”. So the partner translated this as (2*3+5)-(4*14-10).

In some cases the fraction is not well expressed. For example to translate the expression $\frac{1+3(5-1)}{4-(3-1)}$ one student wrote “I add 1 and 3 times the difference between 5 and 1. Then I trace a line under the whole expression and I write 4 minus the difference between 3 and 1”. The partner translated this as

$$\frac{1+3(5-1)}{}4-(3-1).$$

Comparison of the two tables shows no meaningful improvement in the students’ answers: in the final test there are still many inconsistencies between the given linear expression and the translated expression. Nevertheless, we need to consider that the expressions given in the final test are more difficult than those in the initial test. Moreover, we have observed an improvement in the language used by students to translate expressions into natural language.

The translation does not follow the linear writing of the expression but seems to contain, at least partially, structural elements of the expression indicating the main operation of the expression.

Task 2

Students are divided into pairs. One in each pair is given two linear expression like 2*(3-1) to write in natural language and solve. The text is then read to the partner, who translates the statement into a linear expression and solves it. At the end, the pair compares the two results to see if they coincide. The task is then repeated with the roles reversed.

<u>Initial test</u>

Answers	Pairs of students	Observations
Correct answers	2	Stenographic translation

Correct answer in calculus but not in expression translation	3	In general students write (3-1)*2 instead of 2*(3-1)
Incorrect answers	2 (more than 1 exercise out of 4)	Parentheses were not present in the translated expressions, subtraction was not translated correctly
Invalid answer	1	Teacher helped students solve the test

<u>Final test</u>

Answers	Pairs of students	Observations
Correct answers	6	
Incorrect answers	1	
Invalid answer	1	Teacher helped students solve the test

Comparison of the two tables shows improvement in the students' answers between the initial and the final tests. We have observed that even if some students omit parentheses when translating an expression, they nonetheless calculate the result of the linear expression as if the parentheses were present. This interesting result occurred in many cases and calls for further investigation.

Task 3

This task was proposed only in the final test.

One student in each pair is given 3 linear expressions to represent in tree form. These are shown to the partners, who are to translate them into linear expressions.

The table shows the number of correct and incorrect answers for each expression.

Expression	**Construction Tree representation**			**Translation in linear expression**		
	Correct answer	**No answer**	**Incorrect answer**	**Correct answer**	**No answer**	**Incorrect answer**
(2*3+1)+(1/2*3)	6		1 inversion of two members: (1+2*3)+(3*1/2)	3	1	3 mistakes in position parentheses
$\frac{5*(3+2)}{5}+\frac{1}{6}$	5		2 tree shape incorrect; members changed: $\frac{(3+2)*5}{5}+\frac{1}{6}$	5 (the expression $5*(3+2):5+\frac{1}{6}$ is considered correct)	2	

2*(3-1) + 4/2	5	1	1 inversion of two members	3	2	2 inversion of two members
2*(3+5)-4*(14-10)	5		2 tree shape incorrect	4	1	2 tree shape incorrect
$\frac{1+3(5-1)}{4-(3-1)}$	2		5 tree corresponds to the expression $\frac{(1+3)*(5-1)}{4-(3-1)}$	1	1	5 linear expression does not correspond to tree, particularly for parenthesis***
2*5 – (8 – 3)/5	6		1 incorrect tree shape	6	1	
Total	69,4%	2,1%	28,5%	52,3%	19,2%	28,5%

***in this case the answer is considered correct when the linear expression corresponds to the tree representation, not to the initial linear expression

The table clearly shows that at the end of the experiment the majority of students are able to build a tree representation from a linear expression. Translating a tree representation into a linear expression appears to be more difficult. Most difficulties lie in inserting parentheses for translating tree representation correctly. To translate a numerical expression into a tree representation students need to know the syntactical structure of the tree and follow the computational rules. On the contrary, procedural skills are not sufficient to convert a tree representation into a numerical expression, which calls on students to interpret the tree structure.

§2. Relation between students' achievements and the use of the DDA in the context of the PP.

Aplusix allowed students to validate their answers. Comparison of paper and pencil solutions with those mediated by Aplusix led to the emergence of contradictions.

In a-priori analysis we assumed that we would use both types of tree representation provided by the system, namely Mixed Tree and Controlled Tree representation. However, our activities were not centred on the different ways a tree can be built and so during the experiment we decided to use the Mixed tree representation because this gave students more freedom. This aspect was important to us: students should reflect about the structure of numerical expressions, so it was important that they write expressions in leaves and not only numbers.

§3. Relationship between what was envisaged when planning the PPs and the actual results of the TE

In general students' achievements are consistent with what we envisaged a priori. We think that an important reason was that the time schedule and educational goals were well balanced. We had set 8 hours as a reasonable time to learn how to build a tree representation from a given linear expression and, vice-versa, to read a linear expression from a tree representation.

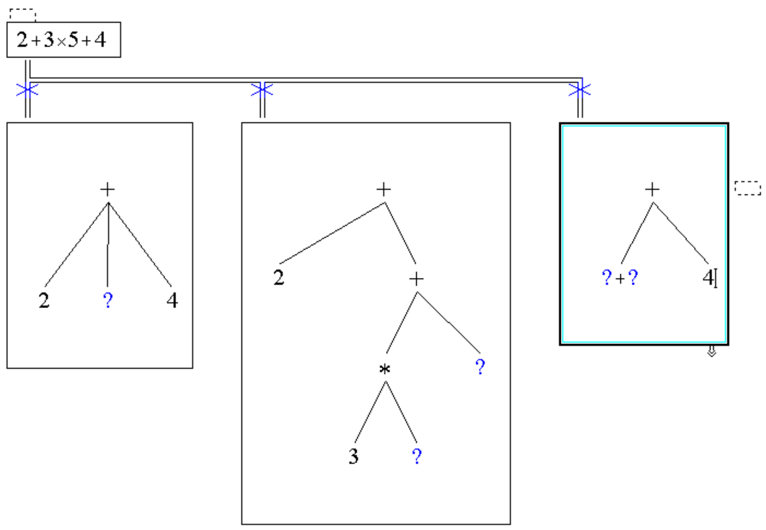
Moreover, Aplusix was a useful tool in reaching our educational goals. The main characteristic of Aplusix is the capability to validate students' answers. Students could work alone and validate their answers using Aplusix whenever they were unsure about their answers.

Nevertheless, even if students are able to express the superficial structure of the expression, they do not quite manage to fully comprehend the systemic structure of a numerical expression. Mistakes concerning the use of parentheses and the priority of some operations over others still emerge in students' protocols and in the final test.

As highlighted by the TE analysis and through analysis of the test results, we have observed that the competencies involved in converting tree representations into linear expressions differ significantly from those involved in converting a linear expression into a tree representation. In particular, procedural competencies proved insufficient for performing the first task. Structural aspects of numerical expressions need to be managed when transforming a tree into a linear expression.

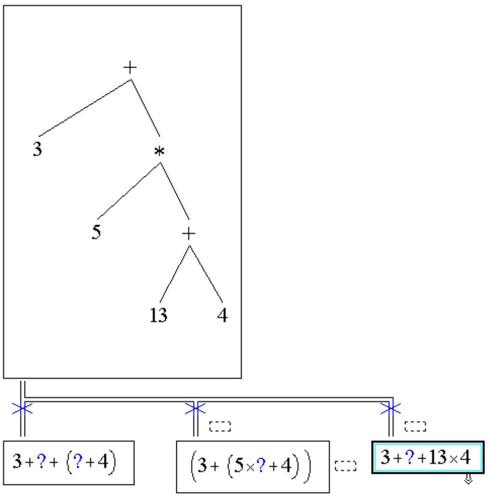
Thus, with hindsight we think that the order of tasks proposed in our PP should be changed. We now see that students ought to have the opportunity to build tree representations of numerical expressions before being called on to build numerical expressions from tree representations, as was the case in the experiment.

Moreover, the task of building different tree representations of a given numerical expression (see the figure below) should be proposed at the beginning of the experiment, rather than at the end.



As a matter of fact this task is cognitively richer than the others because it obliges students to consider the structure of the expression. However, Aplusix provides students with strong support for solving this task and, contrary to expectations, few difficulties emerged in solving it.

Thus, in a future TE other similar tasks might be inserted. For example, a tree could be proposed along with some incomplete numerical expressions, and students could be asked to complete these where possible so that the expression is equivalent to the tree representation. (See figure below).



A.4.11 Synthesis of MeTAH TE with Aplusix

1. Educational goals and students' achievements

Educational goals of our teaching experiment were the following:

The students will be able to (curricular goals):

- identify the form of an algebraic expression given in either of the following representation systems: tree, natural language, symbolic language;
- convert an algebraic expression given in one representation system into another one;
- solve problems involving algebraic expressions given either in natural language or in symbolic language (usual representation).

Pre-test

Students were first administered a pre-test with a few traditional numerical and algebraic exercises such as calculate, factor, develop and simplify, but also tasks requiring to convert algebraic expressions given in usual register into the natural language register and vice versa. The aim of these preliminary activities was one the one hand, to make a diagnosis of students' difficulties in the mastery of algebraic treatment tasks within the usual register, and on the other hand, to let them come to realize the limits of their usual way of “wording” algebraic expressions (two x plus y instead of sum of the product of 2 by x and of y). The results of the pre-test confirmed our hypothesis that even Grade 10 students keep having difficulties in algebra, despite of the fact that in France, the most of algebra is taught in junior high school, i.e., between Grades 6 and 9. Most of these difficulties can be related to the structure of expressions.

<u>Examples of students' errors</u>:

In the class C2, the most errors appear in exercises involving powers and « minus » sign:

$3(-5)^2 \to -3 \times 5^2$; $3(-5)^2 \to \pm 3^2 \times 5^2$; $-5^2 + 7^2 \to 25 + 49$

In the class C1, the same kind of exercises yield also many errors:

$3(-5)^2 \to 3 \pm 25$; $(-3x)^2 \to \pm 3^2 x$; $(3x)^2 \to 3x^2$

But we could observe other kind of errors as well related to the priority of operations:

$2+5*9 \to 7*9$; $2+3x \to 5x$

Typically, this kind of error can be explained by the fact that the pupil does not distinguish between procedural and structural aspects of the expression and performs the operations from left to right.

In the second part, communication games between pairs of pupils, there were surprisingly not too many errors. The majority of errors was due to ignoring parentheses, e.g., the expression a -(x+2) is read "a minus x plus 2". The pupils succeeded the activities in spite of using their usual wording, ambiguous in most cases, thanks to strategies that consisted in reading each particular written symbol, e.g., the expression $\frac{(3x+2)(3x-1)}{a-(x+2)}$ was read "open a parenthesis, three x plus 2, close the parenthesis, open the parenthesis, three x minus 1, close the parenthesis, all this above a minus open the parenthesis, x plus 2, close the parenthesis".

Learning

In the initial PP, the learning sequence comprised 4 phases and was planned to be enacted through six 1-hour sessions:

(ES1.1) Introduction to the tree register;

(ES1.2) Conversion between natural language (RNL) and tree registers (RT):

RNL → RT with Aplusix in controlled mode

RNL → RT with Aplusix in free mode

RT → RNL in paper and pencil

(ES1.3) Conversion between usual (RU) and tree registers (RT):

RU → RT with Aplusix in controlled mode

RU → RT with Aplusix in free mode

RT → RU with Aplusix in free mode

(ES1.4) Treatment in tree register:

Calculate with Aplusix in controlled mode

Develop and simplify with Aplusix in controlled mode

In the class C1, for the institutional reasons, the pedagogical plan was radically shortened: all tasks that should have been done in free mode were not proposed, the paper-and-pencil conversion activity was given as a homework and the whole unit with treatment tasks in tree register was suppressed. Thus, the whole plan was implemented within three sessions.

In conversion tasks done with Aplusix, there were very few errors observed, as we can see in the tables below.

Pair number	Number of correct answers	Number of incorrect answers	Number of non treated exercises
1	7	1	4
2	10	0	2
3	12	0	0
4	6	0	6
5	5	1	6
6	10	0	2
7	10	0	2
8	12	0	0
9	11	0	1
10	10	0	2
11	9	1	2
12	5	2	5

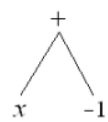
Table 1. C1 class: results to the RNL → RT conversion task (12 exercises).

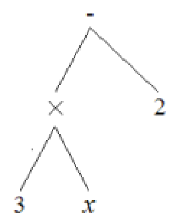
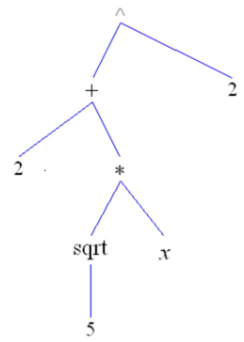
Pair number	Number of correct answers	Number of incorrect answers	Number of non treated exercises
1	8	0	0
2	3	0	5
3	6	2 (mixed repr.)	0
4	5	3 (mixed repr.)	0
5	8	0	0
6	8	0	0
7	7	1 (mixed repr.)	0

Table 2. C1 class: results to the RU → RT conversion task (8 exercises).

As regards the strategies used by the pupils to build a tree, trial and error strategies were the most frequent benefiting from the feedback provided in the controlled mode.

Contrary to the results obtained with Aplusix, the RT → RNL conversion task done in paper and pencil environment revealed many pupils' difficulties with describing a tree representing a given algebraic expression. Several erroneous strategies appeared:

- left-to-right reading, e.g. the tree in Fig.1 is read "x sum of minus 1"	 **Figure 1.**

- starting with the simpler branch, independently from the operator, e.g. the tree in Fig. 2 is read “difference between 2 and product of 3 by x”	 **Figure 2.**
- juxtaposing branches, e.g. the tree in Fig.3 is read “square of two, sum of two, product of x and the square root of 5”	 **Figure 3.**

On the one hand, we notice that the students have acquired correct mathematical language. However, the strategies described above witness about the lack of understanding the structure of algebraic expressions.

In the C2 class, only one group worked on conversion tasks between the different registers in Aplusix. We note this group G1 and G2 the group which has not benefited from this work. Afterwards, the entire class was assigned a homework proposing conversions from the tree register into natural language (the same as C1 class). The analysis of the homework shows a significant difference between these two groups. The majority of responses in natural language from students in the group G1 are expressed according to the structure of the algebraic expressions (e.g., the sum of x and 1) while the majority of responses in group G2 are expressed according to the structure of the oral register (e.g., x plus 1).

	Structural register	Oral register
G1 (15 students)	10	5
G2 (15 students)	3	12

Table 3. C2 class: results to the RT → RNL conversion task .

2. Students' achievements and the use of the DDA in the context of the PP

The results presented above show clearly that Aplusix tasks posed much less, if any, difficulty, since the students worked only in the controlled mode and thus benefited from the feedback provided by the system that allowed them to succeed in solving the tasks proceeding by trial and error rather than analysing the structure of the algebraic expressions. However, as we can see, the success in solving conversion tasks with the system in controlled mode does not guarantee that the knowledge aimed at was used in the solving process.

3. Relationship between what was envisaged when planning the PPs and the actual results of the TE

In C1 class, the students' achievements are not consistent with what we envisaged a priori. In comparison with the achievements of C2class students who benefited from an extended work on the tree register with Aplusix, we suppose that the main reason for the failure of the teaching experiment in C1 class is that is has become too ambitious after having been radically shortened. Indeed, in only 2 hours, the students were supposed to get familiar with a brand new register for representing algebraic expressions, with the tree module in Aplusix, with the mathematical language, learn how to solve conversion tasks between three different registers...

A.4.12 Synthesis of MeTAH TE with Alnuset

1. Educational goals and students' achievements

Let us remind that the main educational goals of the pedagogical scenario was that the students construct:

1. the meaning of function as a relationship between dependent and independent variables;
2. the meaning of the notions of equation and inequations as statements that are true for some values of a variable;
3. the meaning of equivalence between expressions as statements that are true for all values of the variable;
4. the meaning of a solution of an equation as a value of the variable for which the equation is true.

The scenario was implemented in one Grade 10 class in a private high school. Initially, two 1-hour sessions were planned, but the second session could be extended to 2 hours. Both sessions took place in a computer lab, students working in pairs. During the first session, the students were split in two groups of 20 and 14 students respectively, the second session took place with the whole class.

The following report concerns only the first educational goal related to the notion of function[3]. This notion was addressed in the first session whose aim was twofold:

1. familiarisation with Alnuset;
2. studying two functions prescribed by the French curriculum: $x \to x^2$ and $x \to 1/x$.

Regarding these two functions, the students were first asked to observe the relationship of dependence between x and x^2 by noticing that, on the one hand, when x moves on the algebraic line, x^2 (or $1/x$) moves accordingly and, on the other hand, x^2 (or $1/x$) cannot by dragged with the mouse. Then, their attention was drawn to the way x^2 (or $1/x$) moves when x moves on the line. The aim of this task was to develop an instrumental technique allowing to determine variations of a function. This technique is based on observation of the movement of

[3] The analysis of the rest of the experiment is in progress.

$f(x)$ when x is dragged along the algebraic line: when x and $f(x)$ move in the same direction, the function f is increasing, when they move in opposite directions, f is decreasing.

In the sequel, the main results are reported. The analysis is based on the data we collected, namely students' written productions, discussions of a few pairs of students who were audio recorded, and observers' notes.

As far as the first educational goal is concerned, the dependence of x^2 on x was easily perceived by the students due to the dynamic representation of x and x^2 on the algebraic line. Follow some of the students' answers to the questions "What happens to x^2 when you drag x" and "Can you drag x^2 with the muse?":

> "*x^2 moves depending on x (*in French*, x^2 bouge en fonction de x)*"
>
> "*we cannot drag x^2 with the mouse because x^2 depends on x, therefore we have to touch x to make x^2 move*".

However, some students were not satisfied with such observation and they tried to qualitatively characterize the relation of dependence between x and x^2:

> "*x^2 is proportional to x*"
>
> "*x^2 is going farther*"
>
> "*when x is on the negatives, x^2 is always positive, a square is always positive*".

This behaviour manifests the difficulty for the students to consider the general notion of function, they are looking for a specific example, such as linear function, affine function or other.

2. Students' achievements and the use of the DDA in the context of the PP

The results described above show that the dynamic representation of expressions on the algebraic line of Alnuset can contribute to perceiving the dynamic functional relationship between two variables, which is necessary (but not sufficient) to grasp the notion of function. The experimentation shows also that the feedbacks coming from the tool are not easily interpreted in terms of mathematical properties of objects that are manipulated. This emphasizes the importance of the role of a teacher in managing students' instrumental genesis intertwined with the targeted mathematical knowledge.

3. Relationship between what was envisaged when planning the PPs and the actual results of the TE

Students' achievements as regards the notion of function as a dynamic relationship between two variables are consistent with what we envisaged a priori. Indeed, we observed the expected behaviour in most of the students and their written arguments also show that the functional relationship was perceived.

However, as regards the study of the functions, and more specifically their variations, only 2 pairs of students out of 17 succeeded in this task. Most of the students failed to interpret mathematically their observations and these remained at a level of a description what they saw on the screen:

> "*when x is positive, x^2 moves to the right, when it's negative, it moves to the left*"
>
> "*x^2 goes farther and farther*"
>
> "*x^2 never goes under zero*"

“x^2 goes until zero, then goes to the right”

Several hypotheses can explain these results:

- The question asked to the students was very vague. The students did not know what they were expected to observe. Thus, their answers can seem legitimate. However, the following question, which asked directly to deduce variations of the functions from their observations, indicated more clearly what kind of observations were expected.
- The notion of variation of a function was not understood by the students. Many students asked for explanations as regards this notion.
- The instrumental technique for exploring variations of a function is significantly different from the techniques students were taught. Two techniques are used to study variations of a function in a Grade 10: one is based on the “reading” of variations from the curve that is a graphical representation of the function (i.e., the function is increasing on an interval I if the curve is “going up” on I, the function is decreasing on I if the curve is “going down” on I); the other is based on comparing f(a) and f(b) given two abscissas a and b from I such that a<b: f is increasing on I if f(a)<f(b), f is decreasing on I if f(a)>f(b). The representation of a function on the algebraic line in Alnuset consists of a unique pair of x and f(x), which represents any pre-image and its image. This representation does not allow a direct comparison of two images by a function: one has to imagine that the movement of x generates another pair (x, f(x)), interpret the movement of x to the right in terms of increasing the value of x, observe in which direction f(x) moves and interpret the movement of f(x) in the same direction in terms of an increase and the movement in the opposite direction in terms of a decrease. At this moment, it would have been appropriate to link the representation of the function on the algebraic line with its graphical representation by means of Cartesian Plane component of Alnuset, which would perhaps help the students observe the link between a horizontal displacement of f(x) and its displacement on the curve representing the function. In the experiment, Cartesian plane was introduced later, after having worked on functional equations and inequations of the type f(x)=k, , f(x)>k, f(x)=g(x)…. This choice, although consistent with our hypothesis that in order to conceptualise the notion of function, it has to be dissociated from its graphical representation, turns out as not being well judged.

A.4.13 Synthesis of Unisi TE with Aplusix

1. Students' achievements

1a. Achievements with respect to the a-priori envisaged educational goals

The PP is based on the use of the Aplusix DDA and has proposed an introduction to structural aspects of algebraic thinking through the manipulation of numerical expressions. Our educational choices has been motivated by the epistemological assumption of considering algebraic calculation not as a generalization of arithmetical computation but as manipulations based on the equivalence. As a consequence we aimed at promoting a structural approach also in arithmetic. The possibility of identifying a structure in a numerical expression has been envisaged as supported by the innovative representation given by the software, the tree representation (TR), which has been used together with the standard representation (SR), and the natural language (NL). The PP has been implemented in two 9th grade classes (age 14), in a period of about 3 months (around 18 hours each class).

The general educational goals of the PP are:

1. Anticipating the introduction to the algebraic calculation, as a manipulation based on the equivalence.
2. Introducing to the "***structure sense***"[4] of an expression.

Within such global aims, we pointed out more specific educational goals that focus on numerical expressions in the perspective of introducing algebraic calculation. They are:

1'. acquiring the general notion of equivalence between expressions;

2'. acquiring the structure sense for numerical expressions.

In particular, the role played by the properties of the operations to derive the equivalence between expressions is considered a key point in the delicate passage from arithmetical to algebraic computations.

On the one hand, exploiting the potentialities of the TR provides students with the possibility of revising notions already encountered in the previous school level (reinforcement of syntactical skills); on the other hand, the innovative representation combined with SR and NL supports students in the passage from the procedural approach to the structural approach. From a cognitive viewpoint, the recourse to a novel representation (TR), which has different utilization's schemes by those usually adopted (SR), should help to break automatisms linked to the computation of numerical expressions and allows students to focus on the procedures they are activating. The subsequent objectification (Radford, 2003) of these procedures constitutes the basis on which the mathematical meaning of algebraic computation can be developed (cognitive goals). In the meantime, allowing students to become conscious of their own procedures also contributes to pursue meta-cognitive goals. In fact, such recourse should also allow students to gain control competences: they should become able to activate an instrument (the TR) that endows them of a resolution scheme, whenever an expression or a sub-expression results difficult to be treated (tree representation is expected to function as a scaffolding).

The PP was based on the following hypotheses that link the use of the DDA with the Educational Goals:

1. the tree representation provided by Aplusix is a vehicle for supporting the structural sense of an expression. In particular, according to our theoretical framework, TR may be exploited by the teacher as a tool of semiotic mediation for making students acquire a structural sense of expressions (ed. goal 2 and 2');
2. the presence in Aplusix of different kinds of representation systems is effective for the envisaged educational goals. The potential of treatments in tree representation outlines a crucial aspect of algebra: the sense of structure (ed. goal 2). The activities of conversions between different registers, and in particular between standard representation and natural language, have the goal to make the students conscious of a substantial difference between arithmetic and algebra (ed. goal 1).

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[4] For *structure sense* of an algebraic expression we adopt the definition of Hoch and Dreyfus (2006):

"A student is said to display structure sense for high school algebra if s/he can:
- Recognise a familiar structure in its simplest form.
- Deal with a compound term as a single entity, and through an appropriate substitution recognise a familiar structure in a more complex form.
- Choose appropriate manipulations to make best use of a structure."

As a result of our teaching experiments, we can say that most students have learnt:

- the notion of equivalence between expressions (ed. goal 1'); students are able to state the equivalence or not equivalence of two expressions.
- the distinction between a structural and a procedural interpretation of numerical expressions; students are able to recognize numerical expressions having the same structure (ed. goal 2).

To verify and document the achievement of the educational goals from a comprehensive standpoint, we set up a pre-test and final test device. In the pre-test students were required to calculate numerical expressions given in the standard representation, hence their "structural sense" is implicitly tested. Syntactical strategies may provide evidence of procedural versus structural competences in the interpretation of an expression. In the final test, students were explicitly required to provide procedural and structural readings of expressions, and to recognize those expressions having the same structure (ed. goal 2). The final test had the objective to verify whether students have gained competences in reflecting on structural aspects of a numerical expression whatever representation is used. In fact competences are tested both in TR, where, according to our hypothesis the structure is more evident, and in SR, where the structure remains hidden. The following question is a example drawn from the post-test.

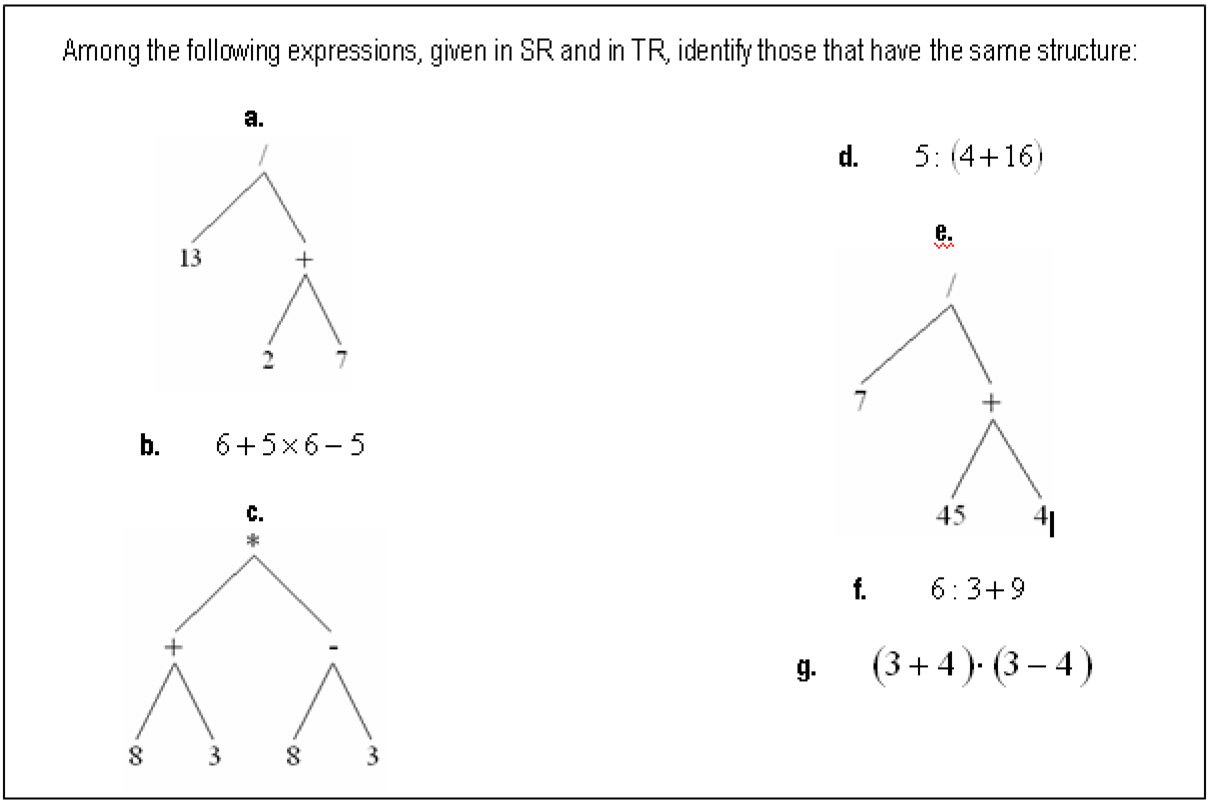


Figure 1

Note that the solution of this task requires to reason in a pure abstract view, that means emptying the expressions of numbers and considering only their structure.

1b. Evidence supporting the claimed achievements

From the results of the final test we can affirm that the majority of students have reached the educational goals.

In the following, a protocol where a student, who correctly solves the task mentioned above (see Fig. 1), shows a structural approach in his solution.

The student state the structural equivalence between a) b) and e) and expresses such equivalence stating the correspondence between expressions given in TR and in SR .

As Fig. 2 shows students' capacities of recognizing a structure even in an string without operands, in spite of the fact that SR is much less effective than TR. We may conjecture that working with trees has functioned as mediator.

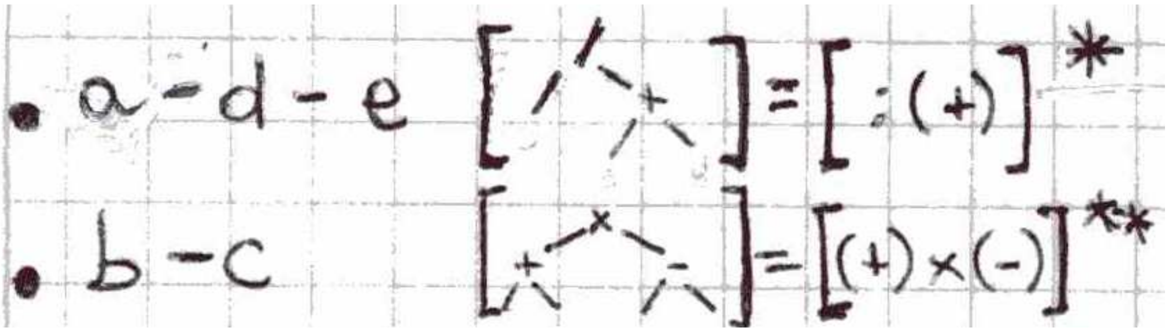


Figure 2

Moreover, the role played by trees in putting into evidence the structure of a numerical expression make it possible for students to resort to TR when they needed 'to see' the structure of an expression. This fact is more evident in some protocols where students show to prefer to resort to TR in order to compute some expressions given in natural language and expressed in a structural way. We can hypothesize that the TR has become an internalized tool. According to our hypotheses the resort to trees could be related to the congruence (Duval) between TR and the structural reading that in this case is reinvested in reverse order: from the reading to the tree.

2. Relationship between students' achievements and the use of the DDA

To describe the role of the DDA with respect to the outlined educational goals, besides the written tests we have monitored the students' production all along the implementation of the PP, by the collection of log files of Aplusix, students' worksheets and written reports, audio and video-recordings, and field notes by teachers and researchers.

By analyzing students' ongoing production with a semiotic lens, we have been able to identify key elements that provide evidence of the role of Aplusix components in students' learning processes, and in the teaching strategy. They have been framed and analysed by means of the Theory of Semiotic Mediation.

We have observed the emergence of some artefact-signs[5], and have managed to identify the semiotic chains that link the emerged artefact-sign to the mathematical meanings. In particular, here after we give an example regarding the notion of equivalence between expressions, starting from the Aplusix feature of feedback. It is taken from the discussion at the end of the Didactical Cycle 1, when the class is discussing about the signs that Aplusix shows during computations:

11. Amalia: Well, there are three signs…well, those two vertical lines are when the passage is right and concluded

[…]

30. Teacher: […] What does it mean "to be right"?

---

[5] An "artefact-sign" is a sign that is directly related to the artefact use in solving the task.

31. Martina: That you didn’t make any mistake in the computations
32. Amalia: That you have not mistaken anything and you can go to the following passage
33. Martina: The computations, the sign…

[…]

54. Teacher: […] And how can we that do not use the computer, understand that things are right without seeing the signs? Why are they right?
55. Ambra: Because if the computation follows a logical thread, it is right
56. Teacher: Because if the computation follows a logical thread, it is right. What does it mean to follow a logical thread?
57. Martina: To do certain operations

[…]

61. Teacher: […] Why are passages right? What does if mean to have the passages right? Where does it lead the logical thread? […]
62. Amalia: Because basically the last passage must give you the result of the first one
63. Teacher: The last passage must give you the result of the first one: what does it mean?
64. Amalia: And yes because basically if you solve the first passage the result must be…equal to the second
65. Teacher: Let's help her to tell it well

[…]

68. Ambra: Yes because finally the result is the simplification of the first, each passage has the same result
69. Teacher: and so?
70. Amalia: Basically, if we have…I don't know…6/3 and we reduce to the minimal terms it comes 2, doesn't it? (The teacher writes on the blackboard 6/3 and 2)
71. Amalia: so I tell that 2 is the result of the first passage

[…]

80. Teacher: […] How can we say that? […] How can we say that the result of 6/3 is 2? In mathematics, when we speak, how can we say that the result of 6/3 is 2?
81. Cora: That the result of 6 divided 3 gives 2
82. Teacher: Yes, but…what do we say of these two (pointing to 6/3 and 2 with the two hands, Fig. 3) here?
83. Valentina: That they are equivalent each other[6]
84. Teacher: That?
85. Valentina: Yes, that they are equivalent one another, they are equivalent
86. Teacher: And what does if mean that they are equivalent?
87. Amalia: That they are equal…

[6] "Si equivalgono"

88. Students: That they have the same value

The teacher starts the discussion by focusing it on the interpretation of the feedback signs of Aplusix. As emerged in the written sheets, at the beginning of the discussion students' assign the meaning of "right passage" to Aplusix symbol ||. According to our theoretical framework, This is an artefact-sign, taking its meaning from the artefact world it is expected to develop towards a mathematical sing referring to the notion of equivalence. During the discussion we can observe the semiotic chain (a sequence of hinged signs) through which the first artefact-sign evolves through the guide of the teacher.:

right / no errors (from line 11 to line 61)

↓

(connecting) passages with the same result (from line 62 to line 80)

↓

they are equivalent each other[7] (lines 83 and 85)

↓

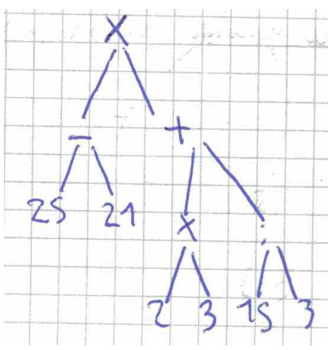
they are equivalent[8] (from line 85)

The semiotic chain come into existence under the constant stimulus of the teacher who asks the students either to make explicit the meanings of the signs involved ("what does it mean", lines 30, 56, 63, 86) or to elaborate on their expressions ("Let's help her to tell it well", line 65; "How can we say that?", lines 80, 82). Note that in this elaboration different signs, be they either belonging to Aplusix, as || or to oral language, are related in a semiotic game generating a complex web of meanings.

By repeating and re-formulating students' contributions on the one hand, and making explicit reference to mathematics language on the other hand, the teacher fosters the weaving of a texture of meanings in which the meaning of equivalence comes to be sided and overlapped to that of right passage. This double interpretation of the Aplusix's feedback signs is the core of the semiotic potential of this specific feature of the DDA in solving the given tasks.

3. What envisaged when planning the PP and the actual results of the TE

Students' achievements are consistent with what we envisaged a priori: most students show to have reached the educational goals and with this respect the DDA has played a certain role, that can be described with the tools provided by the Theory of semiotic mediation.



In particular, it is possible to observe the internalization of the tree representation in some students' productions. Consider the following task belonging to in the delayed test done in paper and pencil.

Two numerical expressions are given in natural language; the verbal expression reflects a structural reading for the first and a procedural reading for the second. For instance: 'the product of

[7] "Si equivalgono"
[8] "Sono ec lenti"

the difference between 25 and 21 and the sum of the product between the product of 2 and 3 with the division between 15 and 3”. Students are asked to compute. In the majority of the cases not only student resort to TR and convert the verbal expression into a Tree (see Fig. 4), but also they accomplish the calculation reducing the levels of the tree, as they were used to do in Aplusix. In other words, beyond showing the accomplishment of an instrumental genesis, it seems possible to claim that the “sub-tree” scheme of use has been internalized.

In fact, the sub-tree scheme, developed within the artefact world, is reinvested in paper and pencil, without a direct use of the artefact.

A.4.14 Synthesis of Unisi TE with Casyopée

§1. Educational goals and students' achievements

The main goals envisaged when designing this Pedagogical Plan were to foster the evolution of students’ personal meanings towards:

1. the mathematical meaning of function as co-variation and thus consolidate (or enrich) the meanings of function they have already appropriated;
2. the mathematical meanings related to the processes characterizing the algebraic modelling of geometrical situation.

More specifically, as for the notion of function, students should consolidate or enrich:

- the meaning of variables both geometrical and numerical,
- the meaning of domain of a variable,
- the meaning of function as co-variation over time (even when different kinds of variables are involved),
- competencies related to the passage between different representations of function (at least, algebraic and graphical ones)

as for the modelling process, students should learn to:

- recognize geometrical variables,
- associate numbers (numerical variables) to geometrical variables,
- associate geometrical variables to numbers (numerical variables),
- pass from not-measurable geometrical objects (e.g. points) to measurable geometrical objects,
- parameterise (optimize the number of variables),
- express the relation between numerical variables through formulas.

In that respect, some remarks are needed:

Remark 1: we remind that according to the designed pedagogical plan students were supposed to students were supposed to have received some formal teaching on functions, variables…

thus as for the notions of “function”, “variable” and related notions, the designed PP was expected to lead students to enrich the meanings they already appropriated.

Remark 2: the teacher was supposed to have some expertise in managing the class activity and in particular orchestrating collective discussions as framed within the theory of semiotic mediation.

Remark 3: we listed above many different specific educational goals in which the main educational goals are articulated. Though all those aspects could be singularly pursued through the planned PP, it is not reasonable to think to be able of pursuing all of them together. Actually, the choice of the specific educational goals to focus on, rests on the teacher. That option certainly depends also on how the activities progress.

§1a. Achievements with respect to the a-priori envisaged ed. goals.

Obviously there is no direct access to the meanings students appropriate. We need some kind of “observables”. For us, consistently with the TF adopted, the observables are the *signs* which students produce and use when accomplishing the assigned tasks.

Thus, the achievement of the envisaged educational goal is attested through the analysis of students’ verbal productions, in particular of the reports students were asked to produce at the end of each session.

Students can be said to have achieved the envisaged educational goals if:

a. **they use specific terms (function; independent, dependent, geometrical, numerical… variable; graph; measure; domain; variation; co-variation; ecc.) in “appropriate ways” (i.e. consistently with their (possible) mathematical meanings, the DDA functionalities and the specific activities at stake)**;
b. **they relate mathematical meanings and processes to the software functionalities;**
c. **they express the main phases characterizing algebraic modelling of geometrical problems.**

§1b. Specification of the evidence supporting the claimed achievements

Evidence of students’ achievement emerges from the analysis of students’ reports, their written solutions to the tasks with the DDA, and the transcripts of the class discussions.

On the one hand, that analysis allows to identify expressions (constructed by students) in which specific terms (see *§1a*) are used to report on the tasks accomplished through the DDA. That witnesses that already formed personal meanings are related to or re-elaborate in the light of the actual use of the DDA (including the specific kind of tasks accomplished through it), thus testifying a progressive enrichment of students’ personal meanings towards the formation of the desired mathematical meanings.

On the other hand, one can identify the use of *artefact-signs*, that is signs referring to the context of the use of the artefact, very often referring to one of its parts and/or to the action accomplished with it. These signs sprout from the activity with the artefact, their meanings are personal and commonly implicit, strictly related to the experience of the subject. But at the same time, those signs have potentialities to evolve towards mathematical signs.

Two “movements” can be attested: the use of already known mathematical terms to describe the activities with the DDA, and the use of artefact-signs in a way consistent with their mathematical potentialities. That confirms the development of a texture of meanings and signs which bridges together the artefact-world and the mathematics-world.

Hereafter there is an excerpt from Valeria's 5th report (homework, after the 2nd class discussion, 5th session)

> What do you mean by the terms "function", "independent variable" and "dependent variable"?
>
> […] The independent variable is the one which is modified first, as a consequence of that the other one [the dependent variable] is modified. […]
>
> Which elements of the software can be put in relationship with those terms? Why?
>
> The independent variable corresponds to the mobile point, because it is the element which can be arbitrarily modified, whereas all the figures […] are dependent variables, because their area and perimeter are modified according to how the mobile point is shifted.

The above excerpt can be analysed at least at two different levels.

On the one hand, we can consider Valeria's answers separately. They are both "consistent" in themselves (though not complete): the former is pertinent to the mathematical meanings at stake, and the latter is pertinent to the DDA functionalities and the tasks accomplished through it. Moreover mathematical signs ("independent variable", "dependent variable") and artefact signs ("mobile point", "figure", "area", "perimeter", "shift", "modify") are consistently used.

On the other hand, if we compare the two answers we can notice an impressive semiotic correspondence between them. Such correspondence reveals the establishment of a consistent relationship between the signs "independent variable" (mathematical sign) and "mobile point" (artefact sign), and "dependent variable" (mathematical sign) and "figure", "area" and "perimeter" (artefact signs), and therefore between the associated meanings.

Finally, from both the answers the meaning of function as co-variation emerges too ("as a consequence", "according to"),.

It is not possible to carry out a so fine-grained semiotic analysis, for every students' written productions. And certainly, there are differences between the students' achievements.

Notwithstanding, we can claim that the envisaged educational goals are at least partly achieved.

As for the notion of function, variables and so on, students use specific terms and relate mathematical meanings to the DDA functionalities in appropriate ways. Though different stages are evident.

As for modelling, the idea of modelling is still related to the actual solution of specific kind of problems. Modelling in itself has not explicitly formulated yet.

§2. Relation between students' achievements and the use of the DDA in the context of the PP.

The Pedagogical Plan is designed consistently with the **Theory of Semiotic Mediation**.

According to this theory, the use of artefacts for accomplishing a task leads the individual to the construction of signs (**artefact-signs**) and personal meanings which are related to the actual use of the artefact. On the other hand, mathematical meanings may be related to the artefact and its use, and mathematical signs can express the relationship between artefact and knowledge.

The evolution of students' personal meanings towards the desired mathematical meanings is fostered by iteration of **didactical cycles**, encompassing the following semiotic activities:

students' activities with the artefacts and pair production of signs, students' individual production of signs, class collective production of signs.

The accomplishment of those activities may lead students to generate artefact signs and personal meanings, related to the actual use of an artefact. Under the guidance of an expert (i.e. the teacher), those signs and meanings may evolve towards mathematical signs and meanings.

Different signs can be identified in the evolution process generating what can be called a *semiotic chain*, that is a chain of signification "in which the external reference is suppressed and yet held there by its place in a gradually shifting signifying chain." (Walkerdine, 1990, p.121).

The PP and the use of Casyopée were designed accordingly.

The following excerpt is drawn from the transcript of the class discussion held in the 5th session. It shows an example of how artefacts signs are produced in relation to the use of the artefact, and how they evolve during the discussion.

5. **Luc**: " you have to choose a mobile point, first […]"

   …

8. **T:** " […] do you see anything similar between the two problems?"
9. **Sam:** "one has always to take a free point which vary, in this case, the areas considered […]"
10. **T:** "Then we have a figure which is…"
11. **Students:** "Mobile."
12. **T:** "Mobile, dynamical. Let us pass to the second phase. Andrea, which is the next phase? […]"
13. **And:** "[…] we need to study that figure and observe what the shift of the variable causes…"
14. **T:** "ok, then? Anybody did that, isn't it?"
15. **Sil:** "[…] by shifting the mobile point one observed as [the sum of the areas] varied"

We can notice:

I. The collective construction of a *semiotic chain*, in which a connection is established between artefact signs ("mobile point") and mathematical signs ("variable"). The elements of this semiotic chain are: "movable point", "free point", "variable", and "movable point". It is worth noticing the two directions: from the artefact sign ("mobile point") to the mathematical sign ("variable") and viceversa. That semiotic chain shows: (a) students' recognition that geometrical objects can be considered (can be treated, can act as) as variables (b) the enrichment of students' meanings of variable to include meanings related to "movement".

II. Elements of a semiotic chain in which the meaning of function as a relation of co-variation of two variables emerges. The elements of this semiotic chain are: "a free point which vary […] the areas" -> "the shift of the variable causes" -> "by shifting the movable point, one observed as [the sum of the areas] varied" … more elements can be found in the continuation of the discussion.

With that respect let us take another excerpt from the same discussion:

74. **T:** "[…] what can one do after that? the third step of that…"

75. **Stud:** “the calculation as a function of the variable”

In the last statement (item 75) three artefact-signs – “calculation”, “function” and “variable” – are related, organized in a consistent[9] way, and condensed to generate a new sign (“the calculation as a function of the variable”) with clear potentialities of evolving towards mathematical signs.

§2a. How the teacher used the DDA

We analyzed the discussion above focusing on the semiotic processes mobilized by students and tried to analyze how they were related to students’ use of the DDA.

The same data can be analysed to investigate the teacher’s use of the DDA: for instance, to analyze whether and how the context of use of the DDA is evoked to foster the evolution of meanings.

For instance the first excerpt shows how the teacher evoked the different problems addressed through the DDA, shifted students’ attention towards figures as dependent variables, stressed the dynamical character of the constructed figures. Those actions fueled students’ discussion which lead to the joint construction of the already highlighted semiotic chains .

That is an example of what we mean by saying that the teacher uses the artifact as a tool of semiotic mediation.

The following excerpt shows an episode in which the teacher did not succeed to exploit the potentialities of Chi’s intervention, who countered “variable” with “variable point”. The teacher did not foster a discussion on that, on the contrary through his intervention he quickly put the question aside.

202. **Lor**: a mobile point on the side […]
203. **Chi**: then, when we had to calculate the area… well meanwhile we put CD as *x*, we set a variable *x*
…
206. **Chi**: we put CD as variable, and not by chance CD, in fact we used a fixed point, C, and a variable point on the segment, D
207. **T:** well, the underpinning idea is to link numbers, and, […] having observed a link between the position of the point D and […] the area of the rectangle […] a link is established between a geometrical world and an algebraic world

That witnesses the difficulty of mobilizing strategies to foster the evolution of students’ signs. In fact the evolution of students’ signs depends on extemporary stimuli asking for a number of decision on the spot.

§3. Relationship between what envisaged when planning the PPs and the actual results of the TE:

Keeping in mind the initial remarks, we can claim that there is a global consistency between what a-priori envisaged and the actual results of the teaching intervention.

[9] Consistent: (a) from the point of view of the DDA functioning; (b) calculation, function and variable are also mathematical terms, their use is also appropriate from a mathematical viewpoint; (c) the statement is relevant to the discussion which is taking place.

Certainly, it clearly emerges that the teacher’s expertise is crucial as envisaged. By “teacher’s expertise” we refer to the teacher’s possible strategies in managing the class activities and especially in orchestrating the discussions as framed within the theory of semiotic mediation. In fact, as previously discussed, there are episodes in which teacher does not succeed to fruitfully exploit the unfolded semiotic potential: the evolution of signs and meanings may take directions which cannot be envisaged because it depends on extemporary stimuli asking for a number of decisions on the spot. Moreover, even when the emerging meanings can be foreseen, it could be impossible to foresee how they will be expressed. Thus one can only plan a plot concerning a general possible path which must be adapted to the actual development of the discussion.

A.5 Teaching Experiment Analysis

A.5.1 Analysis of Didirem TE with Casyopée

Validation of DDA and PP

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

We clearly think that our goals can be reached, but really proving that they have been reached and to provide detailed and objective evidence of this achievement needs some precaution.

Our document posted on WP4'forum May 26th mentions clearly students' achievement first with relation to the goal of understanding the several equivalent expressions of a function goals. But we also express dissimilar achievements with respect to different tasks, reflecting specific cognitive difficulties that Casyopée does not miraculously solve.

"This report does not mean an underachievement with relation to the goal of understanding the several equivalent expressions of a function. This shows that, although these students learnt algebra before and were relatively high achievers, their algebraic knowledge was still weak both with regard to manipulation and to understanding. Actually this insufficient knowledge was challenged by the tasks and clearly they progressed with regard to "completing the square" forms as well as to the expansion (uniqueness). This progress is less visible with regard to factorisation."p.3

Our report also opens on new questions, clearly meaning that more data analysis is needed.

" Teachers' dialogs with students can be described as 'strong mediations' and question the 'a-didacticity' of the situation. Up to what point could Casyopée's feedback make students give up with the global point of view and reflect on the properties of the factored form? Up to what point was this mediation effective? » p.3

After that we provide also evidence of achievement with regard to the goals related to the meaning of a variable and especially the importance of choosing a relevant variable for modelling a functional dependency. This achievement differs from one student to the other. For some students, Casyopée clearly played a role in understanding this meaning, while for others, there is no evidence. We certainly believe that not all students have the same cognitive style and that it can influence how they take advantage of the software. Here also, further analysis would be interesting.

We note also that with respect to the goals relative to the ability to experiment and anticipate in a dynamic geometric situation, to model a geometric situation and to interpret an algebraic result in the geometric context, in depth analysis of students' teamwork show a global achievement, but also diversity among teams.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS? EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

We stress the internal assessment that we choose to claim about students' achievement and about the role of Casyopée and of the pedagogical plan. As a difference with other teams we expressed our goals in terms of learning rather than in terms of students' activity or behaviour, or in terms of classroom functioning, and then it is less easy to provide evidence of achievement.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

- How does the use of Casyopée, especially its extension with the representations of geometric functions, affect the students' understanding of functions?

This general question splits into several ones:

- Q1 : How does the geometrical computation followed by a free exploration of the situation in the dynamical geometry's window of Casyopée affect the students' understanding of the idea of functional dependence in geometrical situations?
- Q2 : How does the computation of dependence implemented in Casyopée (several types of feedbacks linked to variables' choices) affect the students' competency to choose adequate variables in specific geometrical situations? How do the distinct representations of parameters and variables affect the students' understanding of the two notions?
- Q3 : How do the algebraic possibilities provided by Casyopée (e.g. computations in the algebraic window) affect the students' understanding of different representations of the same geometric phenomena?
- Q4 : How to built appropriate situations for exploiting the potential of Casyopée in that respect and what has to be the role of the teacher in the management of these?

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

Our **epistemological sensibility** about functions (Duval, Douady,…) help us to split the analysed situation into several domains of work

- a geometrical frame
- a functional frame which split itself into several registers of representations
  - a graphical register
  - an algebraical register

The DDA has been built in respect of this approach. It is composed of three main windows: a geometrical one, a graphical one and an algebrical one. The hypothesis is that the DDA's use permits students to move easily from one frame to another.

The task organised during the session 6 has been built to manage such movement between the frames and registers. That is why to answer the Re-crqs, we choose in this document to focus on the last session of our TE (session 6) and to analyse students'activity during this session. It is a way to understand if the educational goals have been reached

- EG1 : the meaning of a variable and of a function of one variable (linked to Q1)
- EG2 : distinction between variables and parameters and the meaning of one variable function (Q1 and Q2)
- EG3 : the understanding that a same function may have several algebraic representations (Q3)
- EG4 : the abilities to experiment and anticipate in a dynamic geometric situation, to model a geometric situation and to interpret an algebraic result in the geometric context (Q1, Q2 and Q3)

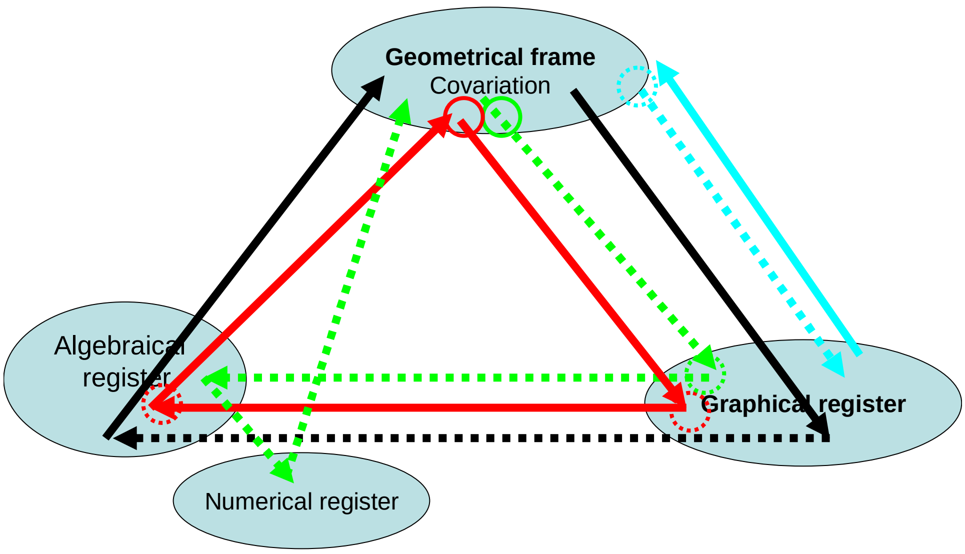
In the document "synthesis of the TE", we gave some answers about the general achievements with respect to theses goals by analysing students'activity during the sessions 1,2, 3, 4 and 5. In the session 6 analysis, we have identified four team's paths during two emblematic episodes linked to our Re-Crq

- E1 : choice of the variable by the students and construction of the function (associated to EG1 and EG2 and more linked to Q1 and Q2)
- E2 : research of the maximum of the function in the graphical or the symbolic window and geometrical interpretation of this maximum (associated to EG3 and EG4 and mainly linked to Q3)

The answer to our last Re-CRQ (Q4) can not be done at this level of analysis. It is too early.

We have identified the path of the students during E1 and E2 looking for the change of worked windows (with the mouse on the screen and the audio tape). Audio tapes permit to mark the helps of the teacher (or the observer) in these paths (coded in dotted lines). Moreover, written productions of the four teams permit to better understand if the EG have been reached.

Green: team 1 ; Red: team 2
Black: team 3 ; Blue: team 4



This schema shows several general results:

- The designed tasks, the scenario and the DDA allow a great diversity of paths among these three poles, according to teams'choices. So the interplay between these settings and registers associated to windows changes may leads to conceptual development.
- The teachers'role is as important as the adidactical potential of the situation in the a posteriori analysis: his interventions have to be analysed. Due to **institutional constraints**, there is a teacher's guidance from geometrical frame → graphical register → algebraic register

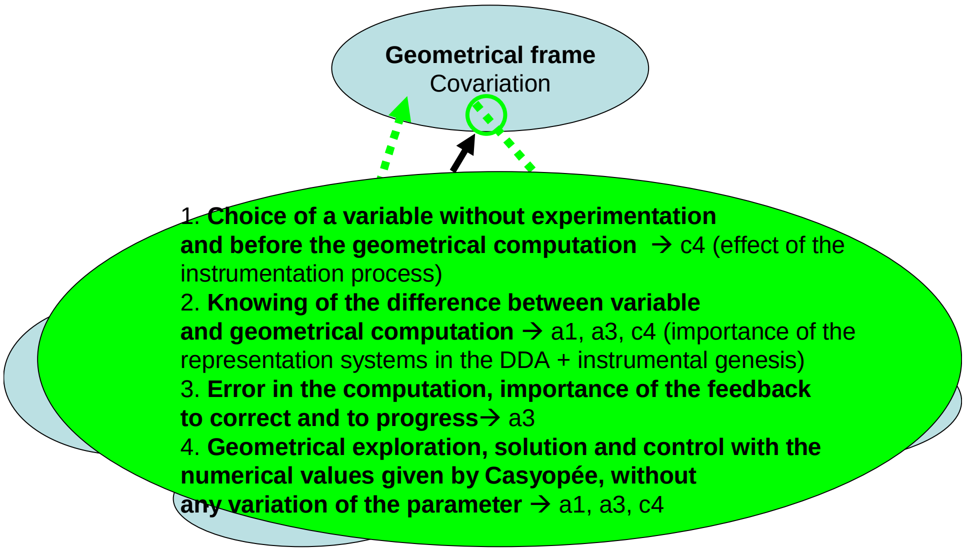
Activity of students can't be understood without considering the whole systemic environment (students, teacher, artefact, institution, cultural concerns…)

About E1: choice of the variable by the students and construction of the function. The choice of an adequate variable and the construction of a function occur in the geometrical frame (geometrical window of Casyopée: creation of a geometrical computation, choice of a variable and creation of the associated function).

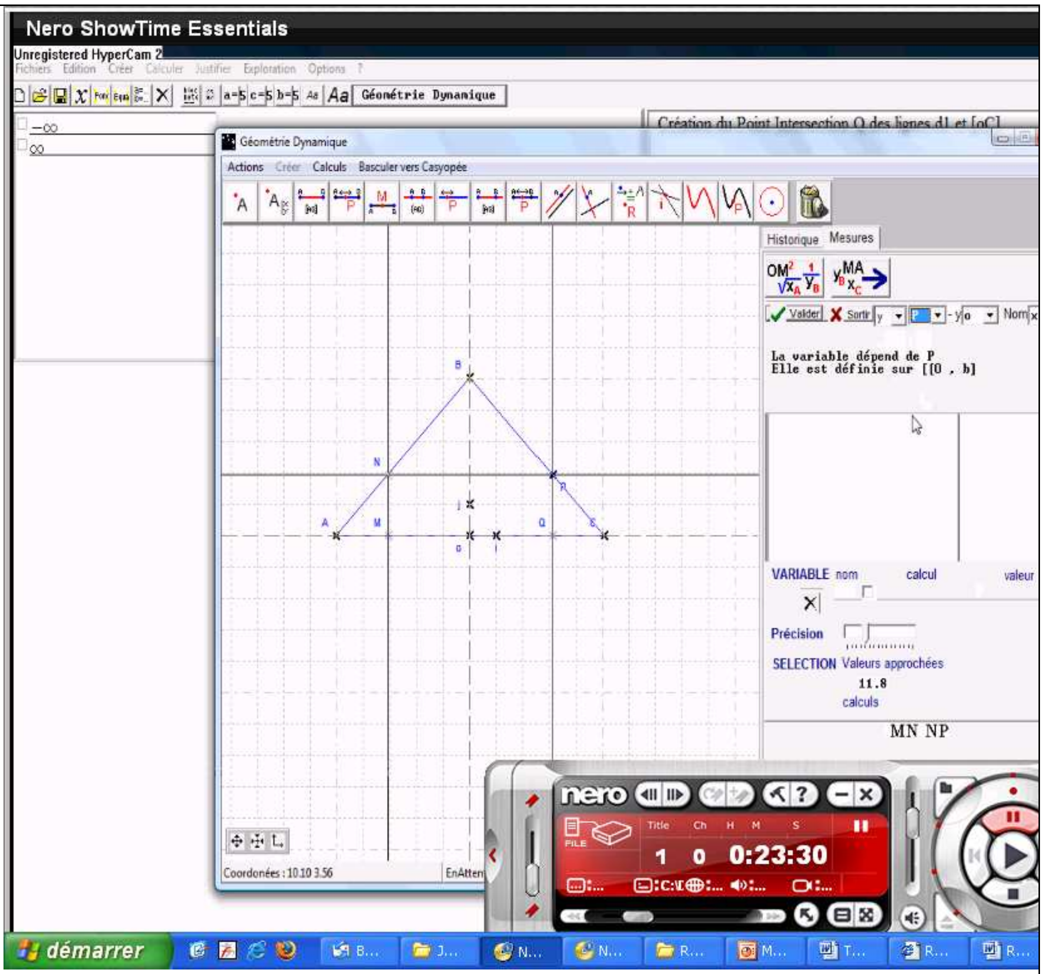
In general, observations show that students have no specific problems to create a geometrical computation which correspond to the numerical quantity they want to study. Nevertheless, sometime they require some teachers' help when choosing the variable and it still exists some specific problems which appear during this phase.

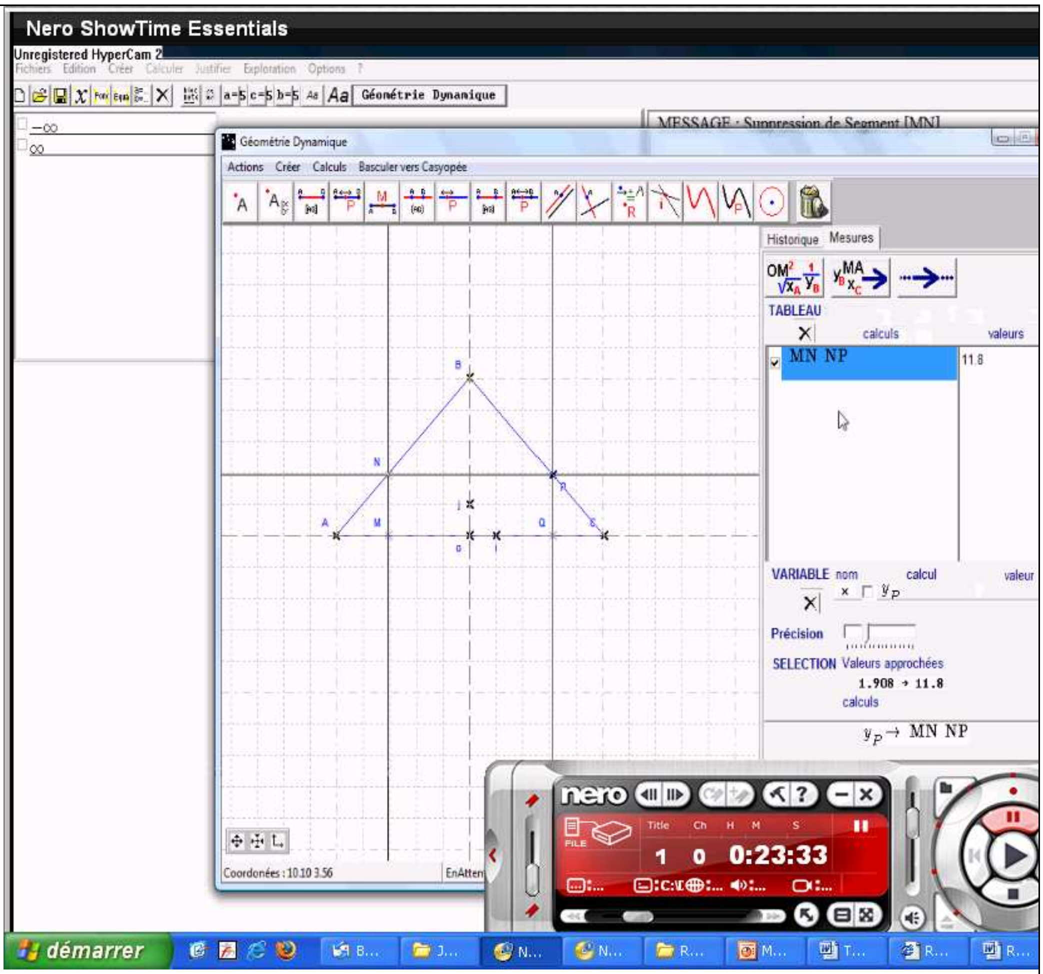
For example, the green team 1 began to choice a variable before experimenting and before any construction of a geometrical computation (without any help). We can interpret this as an instrumentation effect of the previous uses of Casyopée during lessons 4 and 5. Students remember that there is to create a variable corresponding to a free point but they do not create this variable in relation to the geometrical problem they have to solve.

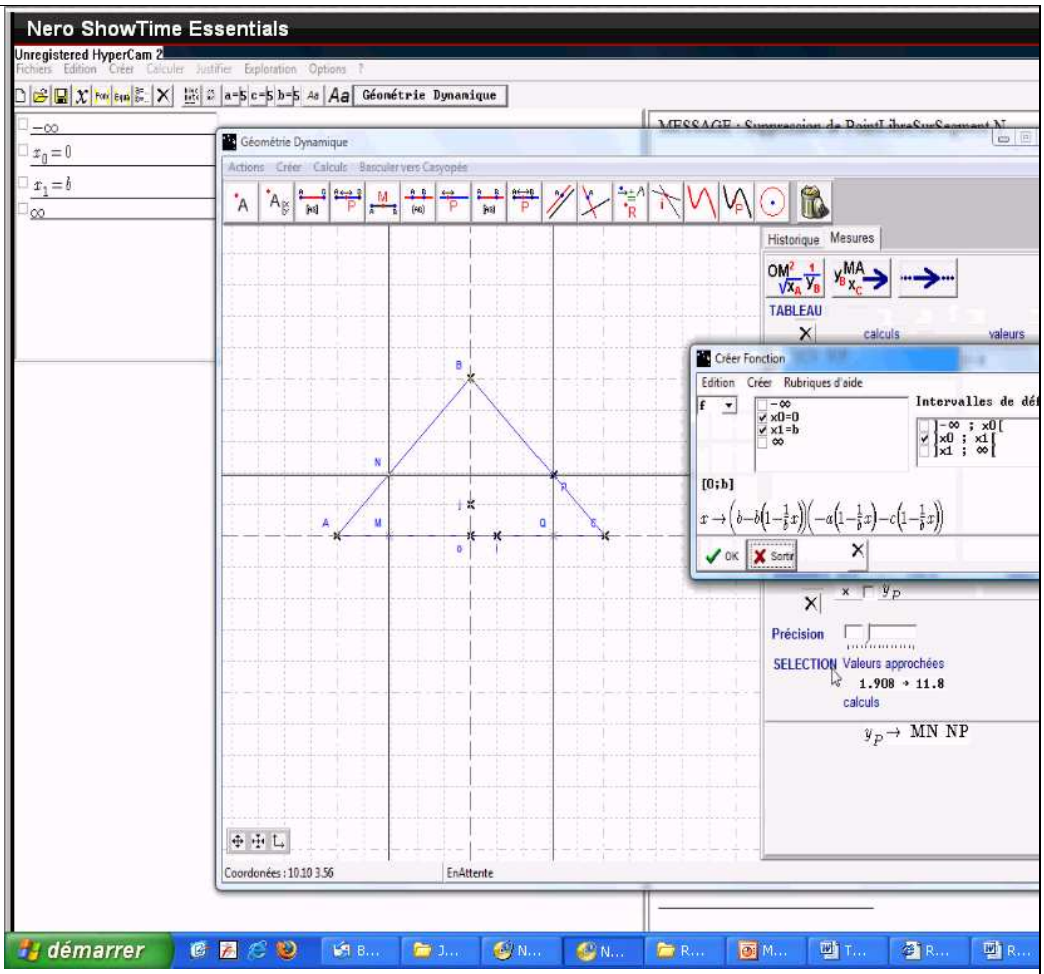
Green : team 1



As another example, the red team still makes confusions between variable and function.

OBSERVER : Move the figure ! **The student move by moving the point P, then they think to chose y_P as a variable** Feedback of Casyopee : "adequate variable" Students click on the button "validate" : *"it is y_P and it's OK, valide !"*	

The button “create a function” appears The students read on the right and the bottom of the window “we have made the **variable** y_P gives MN times NP” Then, the students click on the button “create a function”.	

A new window appears with the algebraic formula. Student 1 says "oulah" and he begins to read the symbolic expression **"b – b…it is with regards to the parameters"**	

At this moment, the observer ask the students "what have you done to get this window?"

"Here"

"And it corresponds to what order ?"

"Create a **variable**, I guess !"

"I is written *create a function*".

There is confusion between variable and function in this team. Nevertheless, the distinction between variable and parameters seems to be understood even especially when one student recognize "**b – b…it is with regards to the parameters**". A specific point we are sensible about is the correspondence between gesture with the mouse and the instrumentation process. For example it seems to be clear that the variable y_P has been chosen in this team because of the displacement the student gave to the point P : **the student move by moving the point P, then they think to chose y_P as a variable.**

As a first answers to Q2, we find

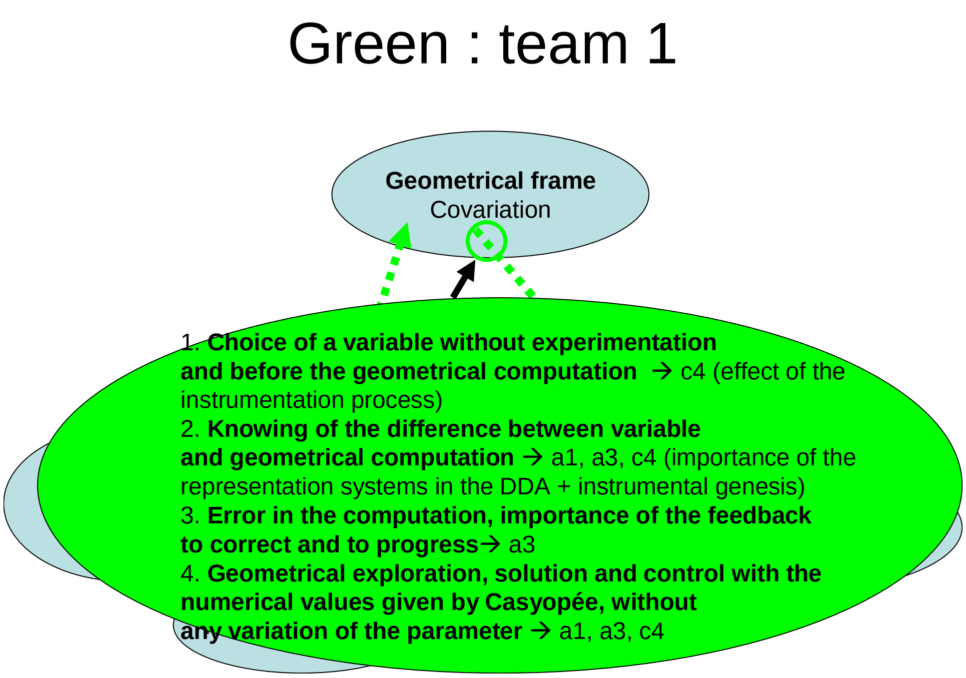
- Students' competency to choose adequate variable →
  - This ability seems to be kept with sometime, some helps of the teacher even if some students choose variable as an answer of the didactical contract, before

creating the geometrical computation and without feeling the necessity of choosing a variable.
  - We have also observed how the choice of a variable can be associated to a gesture on Casyopée. But our task in session 6 permits a too large possibility of variables. Sometime, variables were refused by students only because of some horrible Casyopée algebraic expressions and some students quickly changed of variable. It wasn't an expected feedback.
  - It appears that the construction of a geometrical computation is something easy for students but teacher s' help is sometime useful when students have to find a variable.
- Distinction between variables and parameters →
  - Seem to be good because of the different types of manipulations associated to each of the objects; in general there is no confusion between the manipulations and the meaning of theses manipulations in term of variable and parameters
  - In fact, generally students don't use the possibility to change parameters to question generality of the geometrical situation. There still exists a gap as we also say in the following answers.

About E2 : research of the maximum of the function in the graphical or the symbolic window and geometrical interpretation of this maximum

It seems that many students did not feel the need to use all the symbolic possibilities of Casyopée to find the maximum of the function they have created.

For instance, the green team find the maximum by studying the numerical covariation given by Casyopee (point 4 of the following scheme). That is to say, the move their free point M, see the numerical values of its variations on the screen and pilot this numerical variation in order to reach the maximum numerical (decimal) value given by Casyopée for the geometrical computation.



This observation can be interpreted as a instrumentalisation and an instrumentation interlinked process because Casyopée gives the numerical values and this gift can comfort students that it is not necessary to make a precise study of variations of the functions.

But the task which has been chosen in this session 6 can also be questioned because the function to study is a second degree function whose variations are not surprising. The awaited value of the maximum when values of parameters are integers is a decimal value which is exactly given by Casyopée. So students can have the vision that the study of variation in the symbolic window is not usuful.

This observation has also been done with the read team

Justine : "Donc l'aire maximale, c'est un truc, 12 - 12,45 – 12,5

Lucile : Oui, j'espère environ 12,5 mais avec le paramètre de 5

(…)

Lucile : Il faut une variable mais une variable de quel style ?

(…)

Justine : Peut-on faire un rectangle d'aire maximum ? On va voir l'aire maximale.

Lucile : Oui mais comment on le trouve ? il n'y a pas de tableau de valeurs ?

(…)

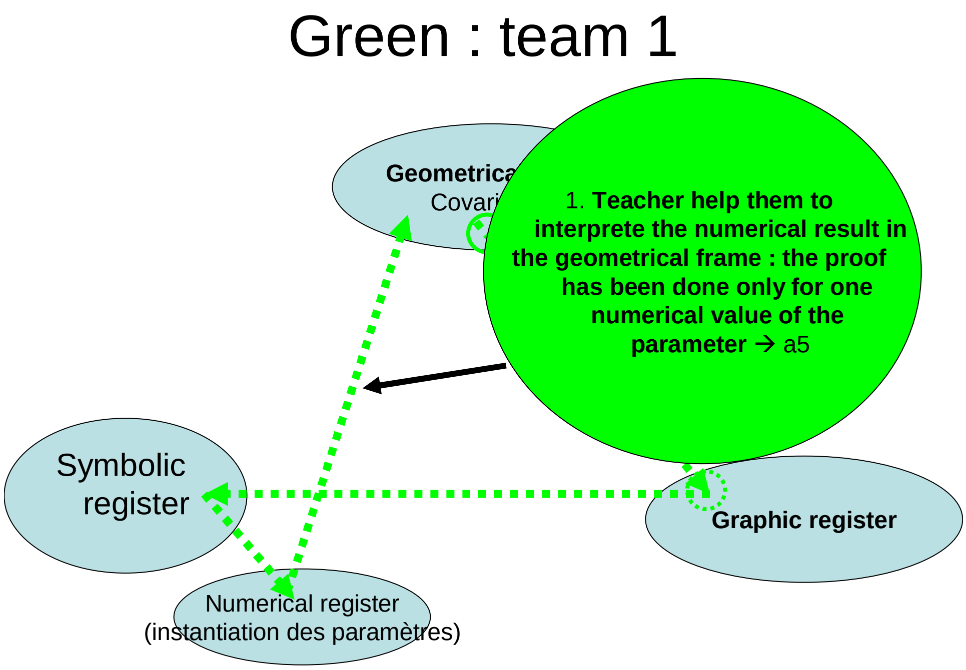
As a second answer to question Q1 , we can say that

- Students'understanding of functional dependence → the dependency is well understood as a geometrical dependency between geometrical punctual values but the link with the global objet « function » in the graphic or algebraic register (curve and algebraic formula) is not done without any help of the teacher. It seems that there still exists a gap and we can ask ourself about the affect of the technology (in general) on such a gap.

As regard Q3, we can say that

- Students'understanding of the differents representations of the same phenomena → it is not so evident for students that a verification of symbolic results can be done in the geometrical frame, the teacher has to help them to make links. Algebraic and graphical manipulations seem to be disconnected from the geometrical computation. In general, students don't move from one system to another by themselves.

For instance this have been observed in the green team path



3. SPECIFY: - THE KIND OF DATA YOU ANALYSED; - THE SPECIFIC ELEMENTS OF OBSERVATION.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ. IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

Said before (change of window on Casyopee associated to change of frame and registers of representation used by students, audio tape to understand what is the help of the teacher in these changes and what are the students processes, written production of students to go further than an "internal" assessment as explained in the document "synthesis of the TE").

After the paris meeting we began to make bimodale transcriptions (gesture / speech) and to find semiotic chains but it still in progress.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

In Paris meeting we had analysed one of the team path making it in correspondance with the concerns we are sensible about

– a1: math objects and interactions in the DDA;
  a3: actions on representations;

a5: interactions between representations within
the DDA and within institutional and cultural systems;

– b1, b2 and b6 : epistemological, semiotic, cultural, institutional concerns in relation with the educational goals;

– c1: tasks and organization;
c3: semiotic issues;
c4: instrumental genesis;
c6: institutional and cultural concerns in the modalities of uses.

- a1, b1 and b2 : sensibility at each step because of the CRQ, the DDA Casyopée and the chosen domain of « functions » with several systems of representations. Our analysis have been done with a split of the mathematic situation according to setting and register of representations coming from epistemological and semiotic knowledge about functions.
- a3 and a5 : sensitivity when students have to move from one system to another, with the help of feedbacks inside the designed « milieu ». Do these feedbacks sufficients to preserve awaited autonomy ? That is the question we have tried to answers as soon as the teachers had to help students and we have tried to find how to manage the situation to permit more autonomy to students ?
- b6 and c6: sensitivity when students reactions are not awaited, the a priori analysis of the activity is linked to cultural and institutional concerns. Sensitivity to explain the type of helps given by the teacher, especially to explain the global path the teacher give to students, from the geometrical window to the graphical window of Casyopée and then to the algebraic window.
- c1 and c3: tasks and organisations are questionned when student's activity is not awaited and/or a teacher's help is needed or given. Tasks and activity are changed by teacher's interventions and we try to analyse what mathematical activity is still in charge of students.
- c4: sensitivity to understand what DDA knowledge has been learned in link to mathematical knowledge, to explain students personal paths through the three poles. For instance the way students can chose the variable according to the point they choose to move on the figure.

A.5.2 Analysis of Didirem TE with Cruislet

Validation of DDA and PP

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

Educational Goals

Use Cruislet's potential for students working on 3D realistic problems enriching the meaning they give to vectors through the use of representations non standard school level and the meaning they give to curves such as circles, spirals or helix through the local generation of these.

These goals were only partially achieved because, in the two experiments, time was short (3 and 2 sessions). Students actually encountered the problems. Some were able to solve them completely while others found the software difficult to use and the tasks very demanding. Time was too short to really intrumentalize Cruislet. Passing to a paper pencil 2D representation to solve a problem of 3D displacement, coordinating Cartesian coordinates and polar representations of vectors ve as well as working on LOGO programs were real obstacles for some.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

Educational Hypothesis

1. The framework TPEs makes possible

the use of Cruislet in the institutional context

the actualization of the original potential it offers for working in 3D geometry […].

2. This requires careful introduction to this complex software through the choice and succession of appropriate tasks, a careful sharing of responsibilities between the students and the teacher, and a careful orchestration of the first phase of the instrumental genesis by the teacher.

We tried to organize the introduction as foreseen in the Educational Hypothesis, but there were obstacles in using Cruislet in the institutional context: the French curriculum leaves little opportunities for not content oriented activities. The TPEs were foreseen as an opportunity, but it was actually difficult to persuade students to choose a project with Cruislet while keeping the TPE's spirit of free choice and open domain. Thus expectations of Hypothesis 1 were not fulfilled. With regard to hypothesis 2, certainly time has been underestimated: most students were not comfortable enough with Cruislet as to carry out a project.

Common Research Questions

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

CRQ for the first experiment (11^{th} grade):

How can the Cruislet representation of 3D displacements be a basis for developing project teamwork?

How do students appropriate and coordinate these representations?

CRQ for the second experiment (9^{th} grade):

Can we design tasks that a priori allow students with a very limited background in terms of vectors make sense of the complex semiotic system offered by Cruislet in such a workshop context, and use it for solving challenging and non trivial tasks involving 3D coordinates, vectors, displacements and curves, far beyond the curricular expectations at that grade?

What specific strategies, use and coordination of semiotic representations emerge from the interaction with Cruislet when trying to solve these tasks?

Remark *The CRQ for the second experiment takes into account:*

- *the impossibility of finding suitable institutional conditions for developing project teamwork*
- *the very limited background relatively to vectors of the students in the second experiment*

2. ANSWER YOUR CRQ.
WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

Regarding the first experiment

- The difficulties met with the first experimentation allow only partial answers to the CRQ (no project teamwork).
- Analysis shows that, in spite of the interest shown by the students for working with the software, instrumentalization of the different representations and the coordination between these required by the piloting of avatars took more time than anticipated:
  - piloting avatars using directions
  - coordinating map and avatar use
  - coordinating direct and programmed piloting
- Analysis of the first experiment also attracts our attention:
  - on the mathematical requirements of the tasks proposed to students in the first phase of instrumentalization (the risk of cognitive overload was certainly under-estimated in the design of the tasks)
  - on the influence of institutional norms and their influence on teachers' decisions even if this specific context of TPE seeming less constrained
  - on the limited opportunity that students have for making sense of the semiotic affordances of Cruislet by the way of a-didactic adaptive processes, in spite of very interesting opportunities

Regarding the second experiment

- Two sessions suggesting:
  - that some main Cruislet features are quickly accessible to grade 9 students
  - the influence on these positive outcomes of the changes introduced in the scenario in terms of tasks and of the tight interaction between the group and collective work along the session
- But also, the same difficulties met like at 11th grade with the design of a flight under constraints requiring the use of some trigonometry and Pythagoras theorem
- The impossibility to getting a precise idea of what has been really learnt

REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY

a) Characteristics of the DDA

Cruislet's attractiveness and affordances for multi-representation have a counterpart: the complexity is very high. There are three ways of navigating: with the mouse, by piloting avatars first by hand, then by LOGO programming. After students learnt to navigate with the mouse, they moved to piloting avatars, but then they could not navigate with the mouse anymore and they were often lost on the chart. That is why they had often to remove and recreate avatars. Then the avatar panel is very complex with several entry boxes some moving the avatar in different ways, and other related to the view (camera properties). It seems that many students do not really master this panel. Exporting to LOGO is done via the same boxes: only a check box controls two very different behaviours of the DDA, piloting an avatar or writing commands, that after execution, will produce the avatar's move. The LOGO panel often confused students because they have no experience of programming,. They for instance had difficulties to insert exported commands at the right place as well as to edit consistently the program.

b) Educational goals

Cruislet vectors did not seem to us a major feature. Thus our Mathematical goals were in relationship with 3D coordinates and trigonometry. These notions are not easy for students and problems with the interface were often mixed with mathematical difficulties, for instance understanding the difference between setpos and setdir had to do with distinguishing points and translations. Difficulties were also a consequence of insufficient ability to represent mentally the third dimension and of lack of method for solving problems in 3D. For instance a student positioned an avatar low above Sparta and wanted to go back to Athena, simply by choosing this town in the list. He repeatedly got the message "Avatar cannot go to this position" because there is a mountain very close to Sparta. He understood that he had to increase the altitude, which he did by trial and error up to 4000 meters, without thinking to go up above Sparta sufficiently high before taking the direction of Athens.

More generally students did not try alone to represent a problem like going from Athens to Sparta in the 2D vertical plane passing by these two towns. After teachers induced them towards this representation they had difficulty to activate their trigonometric knowledge (using atan to find vertical angles).

c) Modalities of use

The tasks we prepared in the pedagogical plan seem a posteriori well adapted for the goals. Nevertheless most students could not achieve them alone and one can be doubtful about what they actually learnt. A minority of students were more active and would deserve further analysis.

Less ambitious tasks (free exploration, trips without constraints…) could have helped students' appropriation of Cruislet, and students could have achieved them alone, but they would not have put actual mathematical knowledge at stake, which is not really acceptable in the French institutional context. More simple tasks would also have been possible by overlooking the geographical background, for instance by making an avatar fly along a horizontal geometrical figure. More ore less consciously, we did not dwell on such tasks, because we thought that they do not exploit the Cruislet's potentialities.

Certainly, a more careful preparation taking into account the instrumental needs of the tasks we prepared would have brought better results. It would have required at least doubling the number of sessions. This again points towards the difficult ecology of this piece of software in the French institutional context.

3. SPECIFY:

   - THE KIND OF DATA YOU ANALYSED;

- The first experiment:
  - Video for the 9 (3x3) sessions observed
  - screen captures for 12 students during the individual or group work sessions observed
  - audio-recording for 5-6 groups for the same sessions
  - successive versions of scenarios, comments by teachers, students' documents
- The second experiment:
  - videos for the 2 sessions observed
  - screen capture for 4 (2x2) groups of students
  - students' documents

- THE SPECIFIC ELEMENTS OF OBSERVATION.

- Successive changes introduced in the design of the sessions by the teachers
- Analysis of videos and teacher mediations
- Analysis of screen captures on specific tasks:
  - the Athens-Sparta trip in sessions 2 and 3 (exp1)
  - the horizontal triangular flight and its vertical adaptation in session 3 (exp1)
  - the landing near Mount Olymp (exp2)
  - the Athens-Corinth flight (exp2)
  - the adaptation of the Logo program for an acrobatic flight (exp2)

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE CRQ.

IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

Answers to the CRQ are evidenced for instance:

- by the distance observed between collective achievements and personal or group achievements
- by the limited use of some representational possibilities (3D controller)
- by the limitations observed to a-didactic functioning

Theoretical frames

- Instrumental approach (especially instrumentalization issues)
- TDS (characteristics of the milieu and of its potential resulting from Cruislet semiotic affordances, notions of didactic contract, of a-didactic interaction/ didactic interaction)
- ATD (institutional constraints, norms) inluencing teachers' decisions before and during the sessions
- The ergonomic-didactic approach of teacher mediations and teacher role and how teacher mediations modify the nature of the students' work anticipated a priori
- Actual realizations that limit the potential for analysis of constructionist perspectives that we anticipated to be especially useful here
- The evident need of an enlarged vision of semiotic representations and mediations (if compared to our usual perspective in terms of semiotic registers) including extra-mathematical representations and sensitive to gestures

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

Usual concerns for us:

1. Semiotic
2. Instrumental
3. Task design
4. Institutional

But a different view and use of these as compared with the Casyopée experiment

Less importance given to epistemological concerns

A.5.3 Analysis of ETL TE with Cruislet

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The educational goals specified in the PP, concerned mainly the concept of function as well as the notion of vector as the displacement in both geographical and spherical coordinate systems. Specifically, the educational goals included:

- The exploration of the concept of function as covariation using the geographical coordinates as a system of reference
- The development of the notions of dependency between airplanes' positions.
- The study of the existence of a rate of change of relative displacements on the 3d space
- The exploration of the notion of the vector as the displacement in Cartesian and in polar coordinate system
- The study the notions of geographical coordinates as the variables of the vector of displacement in Cartesian and/or in polar coordinate system

The criteria we used to specify whether these goals were achieved rely upon the data analysis. To be more specific, in our analysis we searched for meaningful episodes were students get involved with the concepts of function and vector through visual, numeric and symbolic representations. Thus, to articulate the achievement of educational goals, we use episodes where students construct meanings while:

- Using and associating the available representations, (visual, numeric or symbolic).
- Discussing within the groups to accomplish the activities.
- Expressing their thoughts graphically, e.g. making graphs or figures.
- Creating their own activities, based on the "Guess my function" game.

As a conclusion we may say that our educational goals were achieved as far as the concept of function is concerned, as we could see in the analysis. Regarding the concept of vector, we should mention that our initial aim was not for students to study this concept in depth, but rather to use it as a vehicle to displace the airplanes through the use of the corresponding systems of reference. Thus, vector became an object with which students engaged in navigational activities and through this activity, they developed mathematical meanings.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

The hypothesis specified in the TE portrait focused on vector's properties and the ways these can be utilized by students for developing a mathematical language. Specifically, as reported in the TE portrait:

"*While students used the geographical and spherical coordinates develop a more mathematical language based on vectors' properties. They refer to the angles and the length of the vector, to the latitude and longitude. Specifically students distinguish the two systems of reference and make selections about them while navigating in 3d space.*"

Vector's properties and in particular the way a vector is defined in the context of the Cruislet DDA, acted as a vehicle with which students constructed meanings about geographical and spherical coordinates. The criteria specified here are being questioned in SRQ1 and partially in SRQ2.

Common Research Question

2. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

How students using spherical and geographical systems of reference in Cruislet construct meanings about the concept of function?

Note 1: During the analysis process we noticed that the clusters emerged and the episodes used for answering this question can be used for answering the SRQ1 and SRQ2 specified in the TE portrait. Thus, the answers of the following questions are not included in our analysis, as we think that it would be a redundancy.

SRQ1: What kind of meanings do students construct about the relationships between the displacements of avatars while navigating in Cruislet geographical space?

SRQ2: What kind of meanings do student construct concerning the concept of function while making relative displacements in a Cartesian coordinate system?

Note 2: All of the answers given to CRQ and SRQs are a the conjunction of the results of both TEs.

3. ANSWER YOUR SRQ. WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

While students were interacting with the Cruislet environment according to the PP, several meanings emerged regarding the concept of function. We categorise these meanings in clusters that rely upon the concept of function. In particular, there are four major categories:

1. *Domain of numbers:* Students navigating an airplane in the 3d map of Greece realized that the domain of the geographical coordinates is actually a closed group. The 3d map of Greece is a geographical coordinate system with specific borders. The investigation of the range of the geographical borders as the domain of the function became the subject of study and exploration through the use of the DDA functionalities. . In particular, students exploited the two different systems of reference and, experimenting with the values of the geographical coordinates, they define the range of the latitude – longitude values. This specific range of values has been considered as the domain of the functions according to which the displacements of the airplanes are relative. Although students didn't refer to the values as the domain of the function, we interpret their involvement in finding them, as a mathematical activity regarding the

domain of the function.

Students experimented by giving several values to the geographical coordinates of the airplane's position defining at the same time the range of coordinates' values. In the following episode students are trying to find out the reason for not placing the airplane to a given position.

S1: Why?? Mrs it doesn't accept any value. (they gave values in procedure fly1 and the airplane couldn't go).

R: Remember what values has the lat long coordinates?
Lat equal 58 isn't correct? (she also spokes to the next team)

S1: Mrs, it doesn't accept 32 20 100 either.

S2: Greece hasn't value 20 (student from another team speak ironically to him)

S1: Why? The 58 you used was correct?

An interesting issue related to the domain of the function, is that the provided representations, i.e. the result of the airplane's displacement displayed on the screen, helped students realize that the domain of numbers of the two airplanes displaced in relative positions, are strongly dependent. For instance when the first moved to a given position, the second one couldn't go anywhere but the domain of values was restricted by the first position. In the following episode students realized that the 2nd airplane didn't follow them when they fly at a low height. The episode is interesting as it indicates the way students realize the domain of geographical coordinate values that the first airplane can take in relation to the other one.

S1: There are some times that it (meaning the other airplane) can't follow us.

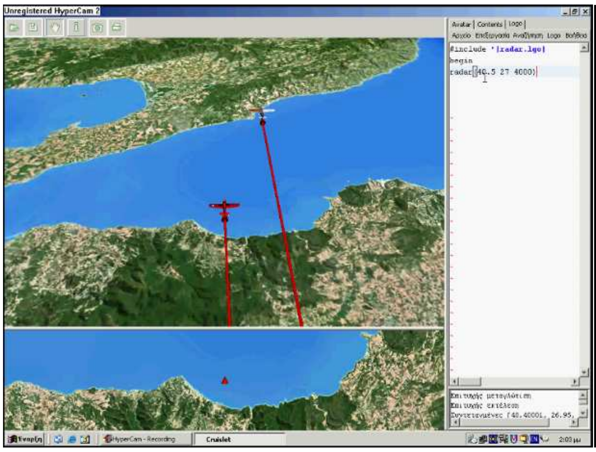
R: Where? When?

S1: When I'm getting into the sea.

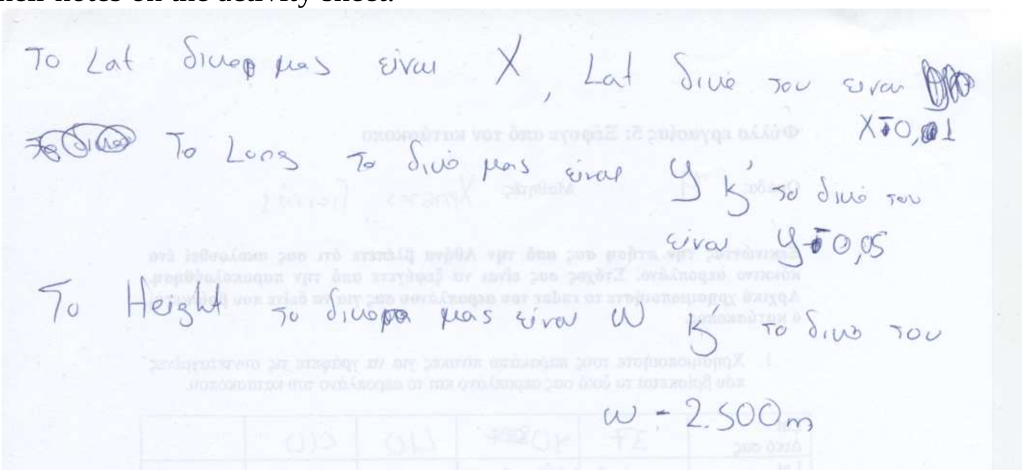
In the language of DF, we could say that the characteristics of the DDA, such as the visualization of the results of the objects' displacements on the map, acted as a mediator in students' engagement with the domain of function. We have to mention that although the modalities of use of the DDA and the communication within the groups didn't reveal that students realized or mentioned anything regarding the concept of function, but rather that they have focused on finding ways to move the airplanes. In other words, students didn't conceive the values of the coordinates as the domain of the function, although they used it in this way. The interpretation of students' actions relies upon our educational goals, resulted to conceive this as a mathematical activity that was related to the notion of function and particularly, its domain.

2. *Function as covariation:* During the implementation of the envisaged PP, students engaged with the notion of function, through their experimentation with the dependent relationship between two airplanes' positions, which was defined by a black – box Logo procedure. Trying to find out the hidden function, students' actions and meanings created suggested they were able to coordinate changes in the direction and the amount of change of the dependent variable in tandem with an imagined change of the independent variable. Our results indicate that students developed covariational reasoning abilities, resulting in viewing the function as covariation.

   Initially most of the students expressed the covariation of the airplanes' positions using verbal descriptions, such as behind, front, left, etc. as they were visualizing the result of the airplanes' displacements. In the following episode students express the dependent relationship while looking at the result displayed on the screen.

	*S1: So, he always wants to be close to us on our left.* *R: Yes.* *S1: And he is beneath, further down to us. Beneath.* *S2: And behind.*

Students experimented by giving several values to geographical coordinates in Logo and formed conjectures about the correlation between airplanes' positions. Through their interaction with the available representations, they successfully found the dependent relation of the function in each coordinate, resulting in their coming into contact with the concept of function as a local dependency. In fact, one of the teams conceived the relationship among each coordinate as a function, as it is obvious in their notes on the activity sheet.


Το Lat δικό μας είναι x, Lat δικό του είναι x-0,01
Το Long το δικό μας είναι y, το δικό του είναι y-0,05
Το Height το δικό μας είναι ω, το δικό του ω-2.500m

Translation

Our Lat is x, his Lat is x – 0.1
Our Long is y and his is y – 0.05
Our Height is ω and his is ω – 2500m.

It
o
hidden functional relationship between the airplanes' height coordinates. In particular, they didn't encounter difficulties in decoding latitude and longitude relationship in contrast to their attempts to find the height dependency. Although all three functions regarding coordinates were linear, students conceived the functional relationship between height mainly as proportional, in contrast to latitude and longitude that were comprehended as linear, from the beginning. In the following episode, students endeavor to apply the rate of change of the function to decode the height relationship. As they were thinking the height coordinates had a proportional relationship, they suggested to carry out a division to find it.

S2: When we go up 1000, he goes up 1000.

R: Do you mean that if we go from 7000 to 8000 he goes from… let's say 2500 to 3500.
S2: He is at… 3000. No. Give me a moment. At 8000 he was at 5500. At 7000 he was at 4500. At 5000 he is as 2500. And then….
S1: We could do the division to see the rate.

An interesting example was the cases of the variation of the height of the airplane every time they pushed the button 'go' in spherical coordinates, when they wanted to make a vertical displacement. In particular, by defining the vector of a vertical upward displacement, students observed that height was the only element that changed in the position of the displacement. Through a number of identical displacements students identified and expressed verbally, symbolically and graphically the interdependency between direction functionality and the height of the airplane. Students' reasoning: "*the more times we push the button GO the higher the airplane goes*", suggests that students developed a covariational reasoning ability similar to the second level proposed by Carlson et al (2001) of how the variables are changing with respect to each other. Moreover, the retrospective symbolic type developed by students (h2= h1+1000) indicates that they realized that the rate of change of the height is constant. In the following figures we can see the result displayed on the screen (figure 1) as well as students' writings on the activity sheet (figure 2).

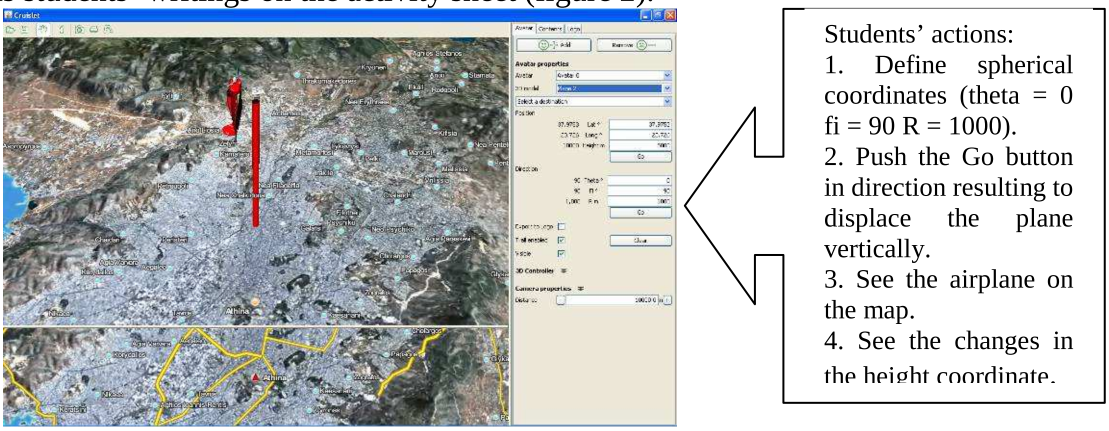


Figure 1

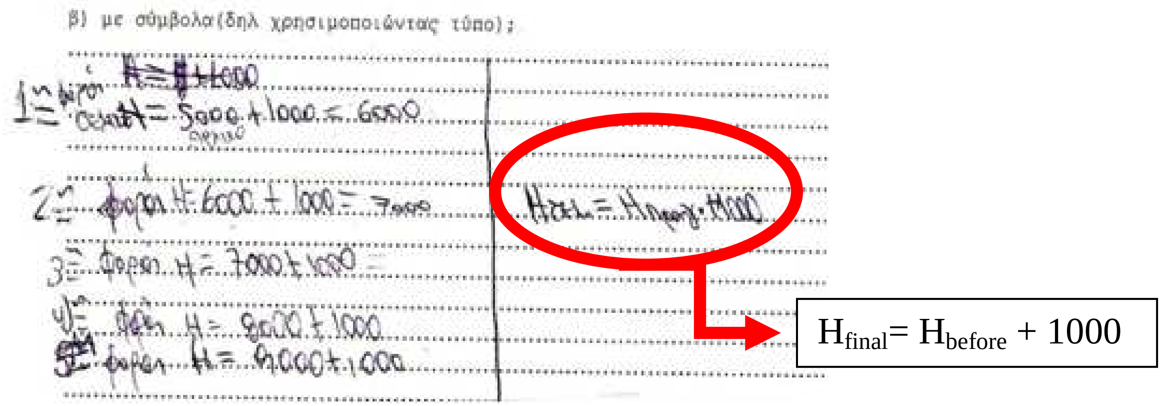
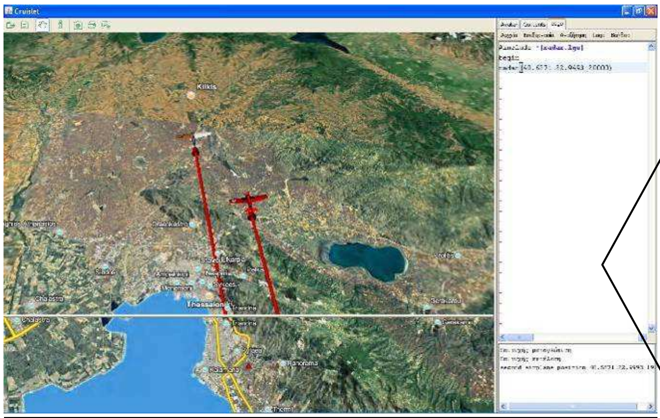


Figure 2

In the language of DF, we could say that the DDA characteristics became a vehicle to engage students with concepts related to the concept of function and their expression in a mathematical way. The result of airplanes' displacements on the screen, gave them the chance to realize the dependent relation in 'visual terms' and then express it in mathematical terms. We believe that the results are mainly based on the way these characteristics were used in the context of the PP activity (modalities of use in the language of DF). In particular, the activity was based on the idea of 'Guess my function' game and the dependent relationship, (built in Logo programming language), was hidden at first. Due to this choice, students focused primarily on the observation of the relative displacements and not on the Logo code underneath it. At the same time perceiving the activity as a game has as a result the engagement of students with the activity.

3. *<u>Inverse function</u>:* Almost all of the teams failed to find out that the use of the inverse function was needed to end the game, meaning to displace the first airplane in a way so that the second airplane would land at a specific city (Thessaloniki). The way the covariation of airplanes' position was represented on the map facilitated students in developing the notion of covariation in airplanes' position. In the following figure, we can see the result displayed on the screen when students inserted Thessaloniki's coordinates to displace the airplanes. The distance between the airplanes and the use of the 2d map, helped them realize that the inverse function was needed.



Students' actions:

1. Insert Thessaloniki's coordinates in the Logo procedure.

2. See the result on the screen (both 3d and 2d map).

3. Realise the distance between the position the wish to displace and current

After interacting with the available representations, as mentioned above, students considered the inverse function as the reverse process in a way that the old outputs could became new inputs. Particularly, as students had found the hidden function, they made conjectures concerning the values they have to input. In the following episode, students communicate their thoughts about the way they must find the coordinates' values and particularly the long coordinate. The episode is interesting as it depicts students' vacillation between using the function (adding the value 0.7) and using the inverse function (abstracting the value 0.7), to compute the long coordinate.

S1: We though contrarily and we added the one that loses in order to go there.

………

R: We want to go to 22. (talking about long coordinate)

S1: Oh, 22. In that case 22,7. (adding the value 0,7)

S2: 21,3. (abstracting the value 0,7)

R: 22,7 or 22,3?

S2: 22,7(students agree)

Their wrong guesses caused misdirection of the airplane. The immediate feedback provided by the DDA encouraged students to think of the concept of inverse function as a process that may be reversed (Carlson, 1996). In the following episode (captured from another team), students manage to end the game by altering the coordinates and using the right values. It's interesting to see not only how they used the DDA representations, but also their enthusiasm while ending the game.

S1: Let's give 38.5.

R: Be careful. We want the airplane to go at lat 39.
S1: 38.5 21.7 10000 (Press insert to run the procedure but the plane displace to another position than they wish to go)
S2: !@#$☠ (Gets upset)
……
S2: Oh, it went at 38 21, but we want it at 39 22 (They give 39,5 22,7 and press insert)
S2: Here it is!! We did it!! Here it is Miss, we did it! YES! Well done to our team! Well done to our team!!!

Although students utilized the provided representations to find the inverse function, they confronted difficulties in expressing their findings in a symbolic way in particular when the got involved with the Logo code, while changing the activity. To be more specific, the coordinates of the first airplane was represented as (:a :b :c) and of the second one as(:a-0.1 :b-0.05 :c-2500) and the condition that defined the end of game related a b c values with Thessaloniki's coordinates. When students tried to change the code, they were thinking that a b c were the coordinates of the second airplane that they were referring to. In the following episode students realize that they should use the inverse function while writing a specific command in the logo code.

S1: Here, a refers to a of the active avatar. (meaning the first airplane)
R: Nice.
S1: Thus a b and c. Thus, given that gayros (the name that students gave to the 2nd airplane) move according to a plus this, in the 'if' command will be a…. as he goes plus to our position. In the if command he has minus.

In the language of the DF we could say that the representations of the DDA supported students in their experimentation and helped them to overcome the difficulties initially occurred. The visualization of the result on the screen, not only did it help them to 'end the game' in the first phase, but to interfere into the logo code and to create something of their own, in the second phase. We conceive these two phases as different modalities of use of the DDA, as in the first phase students perceive and utilize the DDA as a tool to carry out the activity, while in the second phase they perceive it as a tool to create the activity. To sum up we believe that the DDA's functionalities (both representations such as maps, coordinates and the Logo language) acted as a vehicle in different ways of utilizing the DDA.

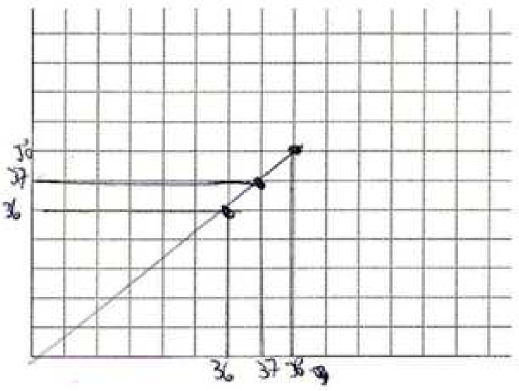
We have to mention that although the modalities of use of the DDA and the communication within the groups didn't reveal that students realized or mentioned anything regarding the inverse of function, but rather that they focused on finding ways to move the airplanes. In other words, students didn't conceive their activity 'as finding the inverse function', although they used it as such. The interpretation of students' actions relies upon our educational goals, resulted to confront this as a mathematical activity that was related to the notion of the inverse function.

4. *Identity function*: Through their interaction with the DDA and while carrying out the activities of the PP, students came in contact with dynamic functional relationships and the way an image of two variables changes simultaneously. During this particular phase of the PP (in the 2nd teaching experiment), the teacher asked the students to reformulate the hidden function in a way that the dependent variable has the same values as the independent. Analyzing the results of students' answers in this additional activity proposed by the teacher, we noticed that they could easily identify and express the concept of identity function either verbally, symbolically or graphically. It's interesting to mention that although students were able to identify the function and

externalize their thoughts in several ways, they didn't mention the phrase 'identity function', as they were not able to 'recall the name'. However this fact didn't prevent them from communicating their understandings about it. The following episode illustrates how students describe the identity function.

S1: We just give coordinates and goes where it has to go.
R: So, which is the function?
S1: I don't know… if we name x the function we take from the position and y that on (referring to the logo procedure) then x=y.

The following figures show how two of the teams draw the graphs in their activity sheets (figures 1 and 2). Figure 1 depicts the way students describe graphically the identity function. From the graph we can see that students used numbers that could be values of geographical coordinates to draw the graph. A similar graph is shown in figure 2, where students from another team preferred to begin their graphs from the origin of the axis and use numbers 1, 2, 3, 4 respectively, as they didn't 'like decimal numbers that geographical coordinates have', as they mentioned. In both figure we espy that students divide both x and y axis in equal segments, meaning that they conceive the rate of changes in airplanes' positions as constant. As a result, we assume that students seem to have realized that changes in independent variable cause congruent changes in the dependent variable.



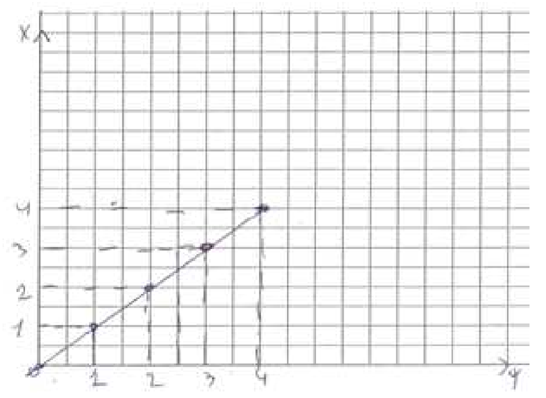


Figure 1 Figure 2

In the language of DF, we could say that DDA representations such as the symbolic representation of geographical coordinates in Logo and the map representation supported the construction of meanings around the concept of identity function. We think that students engagement with symbols like a b and c helped them to realize that coordinates could be represented by a symbol and the dependent relationship between coordinates can be represented in several ways, correspondingly.

4. SPECIFY:
   - THE KIND OF DATA YOU ANALYSED;
   - THE SPECIFIC ELEMENTS OF OBSERVATION.

1. Audio and screen recordings were analyzed verbatim in relation to students' interaction with the environment. We have focused particularly on the process by which implicit mathematical knowledge is constructed during shared student activity. As a result, in our analysis we use students' verbal transcriptions as well as their interaction with the provided representations displayed on the computer screen.
2. Students' activity sheets and notes. This would help us see the way students express their ideas in a symbolic way.

5. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO YOUR SRQ.

   IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

Cruislet was conceived as a digital medium for mathematical driven navigations in 3d large scale spaces. It is based on the idea of multiple linked representations (i.e., any action carried on a specific representation provides immediate change and feedback in all representations, Kaput, 1992). Students interacting with Cruilset environment could define the displacement of the avatar by employing either a Cartesian lat-long-height coordinate system or a vector-differential (fi, theta, r) coordinate system. Students exploiting the provided representations through the use of the DDA functionalities constructed mathematical meanings concerning the concept of function. In particular, during the analysis of the students' constructions concerning the concept of functions we focused on the potential role that Logo programming, mathematical and geographical concepts, relations and representations played.

The tasks which were involved in the PP provided students with the opportunity to explore and generate mathematical meanings, in particular the concept of function, irrespective of the ways in which they might be structured (or fragmented) in the mathematics curricula according to the theoretical construct of the *conceptual fields* (Vergaud, 1990). Our approach to learning promotes also investigation through the design of activities that offer a research framework to investigate purposeful ways that allow children to appreciate the utility of mathematical ideas (Ainley & Pratt, 2002). In this context, the analysis focused on the way that students investigated the mathematical concept of function within a geographical 3D microworld where the foreground issue was the mathematical nature of 3d navigation.

6. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

a) Characteristics of the DDA(s)

a.1 concerns about the ways mathematical objects and their interaction are represented

a.3 concerns about the ways representations can be acted on a.5 concerns about interactions between different representation systems

a.5.1 within the DDA

These concerns guided us in identifying how students interacted with the available representations and the connection between them, to construct mathematical meanings.

b) Educational goals

b.1 epistemological concerns

b.2 semiotic concerns

Guided us in identifying issues regarding the concept of function and the way these are expressed by students in several ways (verbally, graphically, etc.). Particularly, the clusters which emerged from our analysis are based upon different concepts related to function.

Specific Research Question_1

1. REPORT YOUR SRQ.

What kind of choices do students make between spherical and geographical coordinate systems while navigating in geographical space?

2. ANSWER YOUR SRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

Students' interaction with Cruislet environment engaged them with concepts related to the two systems of reference used to navigate in 3d space, geographical and spherical coordinates, as well as with the relationship between them. In this particular question, we endeavor to explore students' choices while using the two systems of reference and the ways these are manipulated in order to navigate in geographical space. Our analysis is based upon students' interaction with the available representations and their preference on one system vis-à-vis the other, while carrying out the PP activities. To be more specific, our analysis revealed three main categories that are shown in the following table:

Category	Geographical coordinates	Spherical coordinates
When? Choice according to the way of navigating.	Displacing at a specific geographical point	Navigating in space

How? Choice according to the representation/ functionality.	Select a destination functionality	3D controller representation
What? Choose among coordinates.	Lat, Long discriminated from Height	Theta, Fi discriminated from R

Although the case for students was to choose among coordinate systems, there were several times that they didn't choose one of them, but rather they tried to create links between the systems of reference, to navigate the airplane. Thus we could add a fourth category on our analysis concerning ways of combining geographical and spherical coordinates.

1. *When? Choice according to the way of navigating.*

Regarding the way of navigation, students preferred to use geographical coordinates to specify a specific position, e.g. a city on the map, in contrast to spherical coordinates used by students to make displacements in space, independently of the destination place, such as figural formations in the air. This was observed in almost all teams, despite the fact that some of them had a strong preference to one system of reference and used it to displace the airplane. In the following episode the teacher asks the class if the 3D controller (the 3D representation of spherical coordinates) is better in any case? Most of the students support this statement in a debate about systems of reference. In the thick of the conversation a student declares that this depends on the situation. The episode is interesting as it depicts students' way of thinking when they had to choose among the available systems of reference.

R: Is Controller better in any case?

S1: Unless we want to go somewhere specific, for instance, at an airport. We won't use 3d controller.

R: Why don't we use the 3D controller to go to an airport?

S1: Because we have to go to the specific airport. If we go with3D controller, we'll go where it lands and we'll crash.

R: Nice. And how do we go to the airport?

S1: We insert its coordinates and it goes. (Meaning geographical coordinates)

A similar situation occurred while another team was trying to displace the airplane in a specific position. In this case students believed that it's difficult to manipulate the airplane with spherical coordinates and that it's 'faster' to use geographical coordinates instead.

S1: The airplane goes faster with position.

S2: Why? We didn't go with the other so as to know.

S1: Yes, but imagine. If we control it with them, we won't be able.

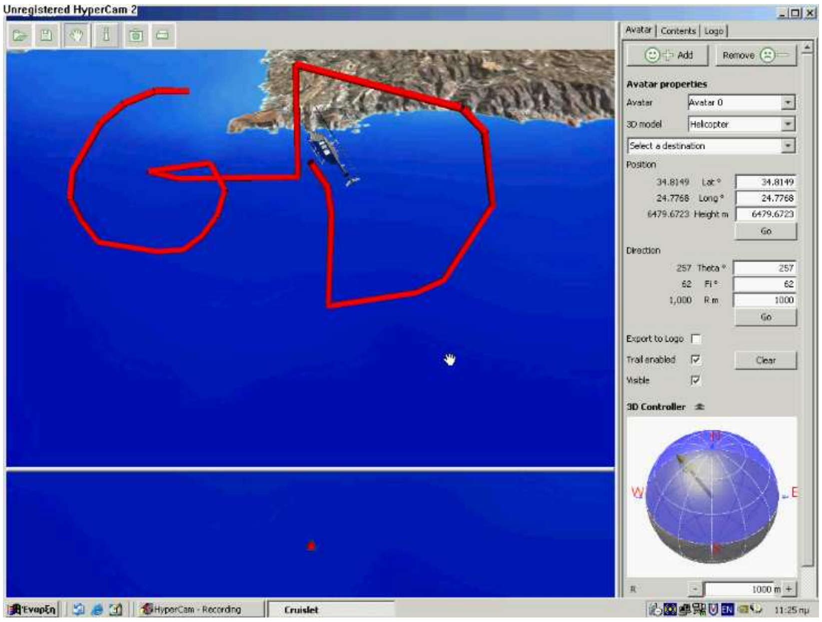
The episode is interesting for another reason as well, as S1 uses the word 'control' to clarify his view of spherical coordinates. This statement is indicative of students' approach, as they viewed spherical coordinates as a way to 'control' the airplane, in contrast to geographical coordinates that displace the airplane in specific places. From our point of view, we interpret this way of viewing systems of reference as an egocentric and an absolute frame of reference, as spherical coordinates has to do with the former and geographical with the later one. As a student pointed out "*The other (meaning geographical coordinates) drives you to an area. I don't believe is as much reliable as direction, because (with direction) you can do changes on your own. Insert values, change meters you want to displace or change the degrees. Anything.".* A more detailed approach can be seen in the following activity sheet, where students support their preference in spherical coordinates.

Το theta και Fi είναι πολύ εύκολο
γιατί μετακινούμε το αντικείμενο προς
όποια κατεύθυνση και όσα μέτρα θέλουμε.

Translation

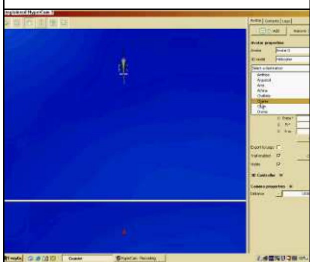
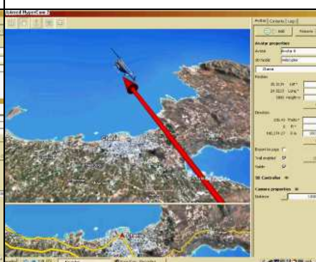
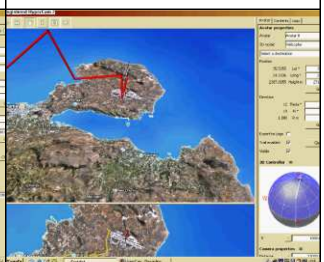
> Theta and Fi is easier, because we displace the object wherever direction we want and whatever meters we want.

As a result of students' approach of systems of reference, they used spherical coordinates when they created figural formations in the air, although this was not included in PP activities. An interesting example is that of a team that decided to draw letters in the air using the 3D controller representation. The following figure shows this construction.



2. *How? Choice according to representation/ functionality.*

The second category concerns the representation used for each system of reference. Regarding geographical coordinates, students preferred to use the ‘Select a destination’ functionality and find the coordinates of the place they wanted to displace the airplane. This is not surprising as this functionality allows them not to search for a destination’s coordinates, but to have it at once. The interesting thing in their actions is that they used this particular functionality, even when the place they chose to go wasn’t included in the list of predefined cities that ‘Select a destination’ had. In this case, they preferred to pick a city near the place they wanted to go, displace the airplane there and finally use coordinates to get the airplane where they wanted. The following table shows the way a team worked in order to go to the airport of Chania.

1st step	2nd step	3rd step	4th step
			
Use ‘Select a destination’ to go to Chania.	Displace the airplane to Chania and see the result both in 3D and 2D maps. The airport is on the right.	See where they want to go by increasing camera’s distance.	Make several displacements to get the airplane to the airport. Use 2d map to identify their position.

Despite the fact that students have been familiarized with Cartesian coordinates in school and several times draw a parallel between Cartesian and geographical coordinates, most of the teams used spherical coordinates to navigate in space and in particular they used the 3D controller representation. Although they were not accustomed to this system of reference, they successfully manipulated the 3D controller in order to displace the airplane. In fact, most of the teams favored this particular representation in relation to others, (such as the editing of spherical coordinates), as they mentioned several times. The following dialogue depicts the way students support their choice in using 3D controller, when the researcher asked them to explain this preference.

S1: 3D controller is easier.

R: Yes? Why?

S1: Because we choose here and then press 'Go' and it goes.

S2: Yes, you have to know where it turns.

R: Why? Instead of using3D controller, why don't you go there and edit the values? Why this is easier to do?

<u>*S1: Because, here we go (moving the arrow of 3D controller). It's more fun and you also play with the arrow. Otherwise, I have to think where it should go...*</u> *Do you understand? Am I clear?*

R: Therefore we work mostly with r, fi, theta when we have the controller. When we don't, we don't work with them.

S1: No. We must think and…

R: Is it difficult, difficult to think? At least have you understand what fi, theta, r are doing?

S1: Hm, we have understood something.

S2: Yes. Yes. We have understood.

After this dialogue, students explain to the researcher their thoughts of spherical coordinates and they seem to have been accustomed to them. Thus, we may consider that their choice don't rely upon their understanding of coordinates, but on a preference on this representation. We consider this episode as illustrative as it underlines two aspects regarding students' thinking of the 3D controller. The first one is that they were thinking of it as '*fun*' as they were able to '*play with the arrow*' and the second one is that they believe that when they use it they 'don't have to think'.

3. *What? Choice among coordinates.*

In this session of analysis, we report students' choices regarding the three coordinates each system composed of and how they were manipulated in order to displace the airplane.

An interesting issue is that students confronted latitude and longitude in a different way as they manipulated height in order to specify a position in space. In particular, most of the times they edited lat and long coordinates up until the airplane was displaced to a specific point of the map and afterwards they were editing the third coordinate, the height. In fact, at their experimentation, many students forgot to edit height as they were concentrated in trying to find latitude and longitude of a place. We could say that may be this is explained by the fact that they were not familiar with the environment and thought that the environment 'reminds' previous positions or coordinates. But this is not the case as such confusion occurred only with height and not with other coordinates, even if one of them remained stable. A possible interpretation about this confusion is that students are accustomed to 2d representations where they manipulate only two magnitudes and this is the reason why they usually preferred to fly at a fixed height. On the other hand if we accept the view of Dalgarno et al. (2002) that we understand 3D models through multiple 2D representations, maybe students had focused subconsciously on a simplified 2D way of visualising the displacements of the airplanes. We have to mention that although students ignored height several times, it was height coordinate that was firstly understood.

Our findings in relation to spherical coordinates are compatible with these on geographical, in the sense that students discriminated Theta and Fi coordinates from R and additionally to the fact that they were accostumed better to the late one. This was not surprising as the measure of these coordinates are different and students identified easily what each one represented. Comparing the manipulation of these spherical to geographical coordinates, we found that in this case, students also focused on changing 2 of the three coordinates (theta, fi) in order to find the right direction. Only afterwards were they editing R, that is the extent of airplane's displacement. In fact, changes in R occured mostly when students had already made a dispacement and from the result displayed in the screen, they could estimate its magnitude easier. We could say that the utilization of R independently of the other coordinates, may rely upon the fact tha they used 3D controller representation most of the times that doesn't have the R coordinate built in.

4. *Create links between geographical and spherical coordinates*

Students didn't always choose one system of reference to navigate in space, but several times combined both to make a displacement. In this way they created links either between distributed coordinates (e.g. height of geographical and fi of spherical) or between all three of coordinates for the two systems of reference.

i. *Links between distributed coordinates.*

In their attempt to place the plane at a specific height, students used primiraly the height coordinate. However, there were some teams that were using spherical coordinates to carry out almost all displacements. Based on students actions on a team like that, students were trying to find a way to raise the airplane's height to a specific value, while utilizing the spherical coordinates. In fact one of them gave the idea to use the fi coordinate and raise the airplane by asking the other one: '*The height is fi?*' and afterwards he edited the fi coordinate's value in order to raise the plane. This statement is interesting as the student endeavor to create meaning around the fi angle that represents airplane's perpendicular angle, in relation to the height that the plane will be placed.

Another episode where students create a link between coordinates is that of longitude and theta coordinates. In the following episode the students of a team argue about the system of refernece that displace the airplane 'right – left'.

S2: It goes right and left. (referring to longitude)

E: Right and left.

S2: Yes.

S1: No. Theta is right and left.

S2: These are the degrees.

S1: Yes, the degrees it turns to the left or right.

S2: I'm saying to displace at the same time.

The episode is interesting as it depicts the way students verbally express the way they realize the displacement while using longitude or theta angle of spherical coordinates. In both cases they use the expression 'right – left' giving the displacement a sense of direction. However, S2 supports that longitude doesn't have to do only with turning like theta, but with displacing as well. The way he externalizes his thought demonstrates that he is aware of the interdependent relationship between longitude and theta.

ii. Links between all three coordinates

The manipulation of 3D controller acted as vehicle with which students realised the notion of vector as the displacement and associated airplanes' displacement with the variation in geographical coordinates. In this way, students explored vectors' properties as they constructed links between geographical coordinates (the variables of the vector of displacement) and the spherical coordinates. In the following episode we can see how the controller is used to identify the dependent relationship between coordinates' values. In particular, the student is using the arrow to prove the way values of geographical coordinates change relatively to the arrow movement.

R: You're saying that coordinates change. (meaning geographical coordinates)

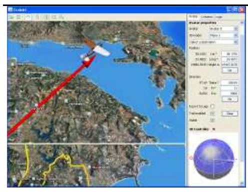
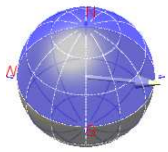
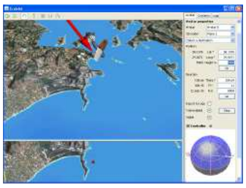
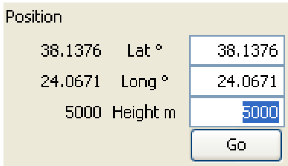
S1: Yes

R: Increase or decrease? What happens?

S1: It depends on where the arrow's direction is. (moving the arrow of the 3D controller)

Another example of controller's utilization to create links between different coordinates, is shown in the following sequence of students' interaction with the environment, where they utilize both spherical and geographical coordinates to specify a position in space.

Cruislet environment	Representation	Students' actions

		Manipulate 3d controller in order to specify direction of displacement.
		Change the height in 5000 meters and displace the airplane by pressing the 'Go' button.

The sequence of students' actions indicates that they endeavour to associate the displacement in 3d space through the use of both systems of reference. Initially they use the 3d controller representation (spherical coordinates) and in this way they specify a point on the map as the geographical coordinates change simultaneously. Their second action includes the setting of one of the geographical coordinates as they want to place the airplane at a specific height on the map. In this case students utilised both Cruislet functionalities and the representations provided, as they attempted to combine the two systems of reference to displace the airplane.

An interesting dialogue that demonstrates the use of the 3D controller representation as a way of combining coordinates is the following one.

R: Why it's better? (meaning the controller)

S: Because it combines both.

R: Which?

S: Because it has, west, north and east and all these, we can do position. And because of the arrow, we can do theta and fi. In other words…

R: You confused me.

S: We can do position because of the North, South, West, East. And with the arrow, we can also do inclination.

In this dialogue S endeavor to support his statement that the 3D controller is the best representation to use. In his attempt to prove this, he is trying to correlate issues regarding both systems of reference, such as geographical directions that are represented on the sphere of the controller, with the arrow that defines the direction of the intended displacement.

As a conclusion, we could say that in the language of DF, students' choices among the different coordinates' systems were based upon the modalities of use of the available representations built in the DDA.

3. SPECIFY:

- THE KIND OF DATA YOU ANALYSED;
- THE SPECIFIC ELEMENTS OF OBSERVATION.

1. Audio and screen recordings were analyzed verbatim in relation to students' interaction with the environment. We have focused particularly on the process by which implicit mathematical knowledge is constructed during shared student activity. As a result, in our analysis we use students' verbal transcriptions as well as their interaction with the provided representations displayed on the computer screen.
2. Students' activity sheets and notes. This would help us see the way students express their ideas in a symbolic way.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO YOUR SRQ.

   IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

Meaningful formalism, constructionism, *half-baked microworlds as well as* the idea of instrumentation and instrumentalization are the theoretical constructs were used in the analysis process. Cruislet microworld is designed to provide students for *instrumentalization* through constructionist activity in the context of *half-baked microworlds* (Kynigos, 1992 and in press). In particular we use the idea of half – baked games. These are games that incorporate an interesting game idea, but they are incomplete by design in order to poke students to finish or change their rules. Thus students explored the Guess my flight game play, changed it and thus adopted both roles of player and designer of the game. From this point of view the work and play with Cruislet is based on the idea of instrumentation and instrumentalization (Guin &Trouche, 1999) since displacement rules questioned and re-defined by the students resulting in a variety of artefacts. In our analysis we focused on those incidents during the teaching experiment where students seemed to be engaged in the process of instrumentation and instrumentalization by exploiting the rules of the Guess my flight game and then by setting their own rules resulting on the development of new games.

On the other hand, the key point here is that students built their models in Cruislet that can act as a support for developing new meanings by investigating their hypothesis and argumentation in social contexts. Displacing avatars and articulating rules of and relationships between the displacements can thus provide an action/notation context which can be a new resource for activity and construction of meanings, not so dependent on the medium for its expression. Noss and Hoyles (1996) introduced the notion of *situated abstraction* to describe how learners construct mathematical ideas by drawing on the linguistic and conceptual resources available for expressing them in a particular computational setting which, in turn, shapes the ways the ideas are expressed. In our analysis, we focused on students' actions within the provided representational contexts (visual, graphical, Logo programming) and systems (geographical and spherical coordinate systems). Students reflecting on these actions expressed their ideas, construct and developed mathematical meanings. We focused on those episodes that students seemed to realise the role of the different representational systems on 3d navigation process and built relationships between them.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

a) Characteristics of the DDA(s)

a.1 concerns about the ways mathematical objects and their interaction are represented

a.3 concerns about the ways representations can be acted on

a.5 concerns about interactions between different representation systems

a.5.1 within the DDA

These concerns guided us in identifying how students interacted with the available representations and the connection between them, to construct mathematical meanings. In particular, we focused on the ways representations could be utilised differently, according to students' interaction between different representational systems.

b) Educational goals

b.1 epistemological concerns

b.2 semiotic concerns

Guided us in identifying issues regarding the concepts of geographical and spherical coordinates.

c) Modalities of use

c.2 concerns about the functions to be given to the DDA and their possible changes

Guided us in identifying the DDA modes of use by students according to the PP.

Specific Research Question_2

1. REPORT YOUR SRQ.

What kind of mathematical meanings do students correlate with geographical concepts while navigating in geographical space of Cruislet environment?

Note: Although this SRQ wasn't included in the TE portrait, from our analysis became obvious that the way students correlate mathematical and geographical concepts constitutes a major part of their interaction with the environment.

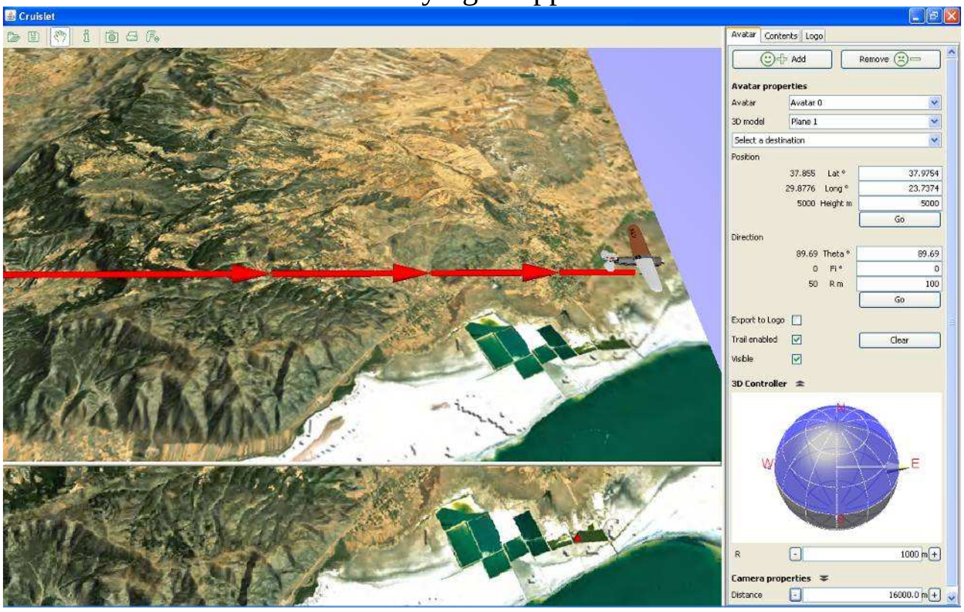
2. ANSWER YOUR SRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

In the Cruislet environment, mathematical concepts are integrated with geospatial representations and information, providing opportunities for processes of mathematisation of geographical space. As a result, the utilization of the available representations supported students' construction of mathematical meanings that are strongly dependent upon geographical concepts. In the following paragraphs we use some illustrative examples of mathematical meanings that students correlated with geography.

1. *Concept of limit*

Students navigating the airplane in 3d space developed an interesting intuitive approach of the concept of limit. Specifically, while approaching a specific point on the map, they used the spherical coordinate system of reference by gradually reducing the measure R of the vector of displacement (see vectors' length in figure below). The following figure depicts students' interaction with the environment while trying to approach the east limit of Greece.



An interesting excerpt is exemplified below, where a student explain their strategy in finding the limit.

S2: We went eastwards until it says that we can't go more. But we could see that we could go a little bit more. We just reduced meters, in a way that it could go till the end.

Students' strategy seemed to be closely related to their idea of the concept of limit as the phrase "*go a little bit more*" suggest. Except the excerpt above, there were several other relative statements captured from other teams such as "*I approach something as near as possible*", "*I had to reduce the step…*" that support our findings. Thus, we could say that the environment provided opportunities to students to approach the concept of limit. We should mention that the mathematical concept of limit underpinning students' strategy rely upon our interpretation of students' actions as they didn't mention anything concerning mathematics and especially limits.

2. *Vectors*

While interacting with Cruislet environment, students defined the vector of displacement and through this activity they got involved with the notion of vector . As a result, several meanings emerged concerning vectors and its properties. In this session we present meanings regarding vectors in relation to geographical concepts.

i. *Vectors' magnitude*

Vectors' magnitude is represented by R in spherical coordinates, so it had to be defined when this system of reference was utilised. During their experimentation students realized that R was remaining constant for a displacement between two specific cities and additionally that was independent of the direction of the displacement. In the following episode students displace the airplane between two cities in their attempt to find their distance.

	Dialogue	**Interaction with DDA / Comments**
S1	*This must be their distance.*	Shows the vector created by airplane's displacement from Arta to Amfissa.
S2	*Yes. But how can we find it?*	
S1	*The R m.*	Meaning R in spherical coordinates.
S2	*No, it's not R m.Oh, you're right! Wait.*	Displace the airplane from Amfissa to Arta and they watch R values in direction.
S1	*You see? It's the same.*	

The interesting issue is that although they displaced the airplane towards one direction, they wanted to verify that the distance was remaining constant for the inverse displacement as well. If fact S1 used this as an evidence to persuade S2 that R represents the distance between the two cities. Our interpretation of S1's way of thinking is that perhaps he used his intuitions or pre-existed knowledge to apply a property of vectors' magnitude in this particular situation.

ii. *Addition of vectors*

An interesting episode was that of a team that used intuitions to identify the resulting displacement if this is defined by multiple displacements. This was occurred while students were trying to construct the rules of a game for the other team. To be more specific, students' idea included the relative displacement of three airplanes, based on planes' coordinates. Here we focus only on the correlation of two planes' displacement (named red and blue by students), as they were moving relatively to theta angle and particularly their dependence can be represented as $\text{Theta}_{\text{blue}} = \text{Theta}_{\text{white}} + 180^{o}$. One of the procoditions of the game was also that the first (white) must go to a particular city (i.e. Thessaloniki) to end the first phase of the game. Initially students sketched their idea in order to explain it to the teacher, as shown in Figure 1. In the following excerpts, the students explain their drawing:

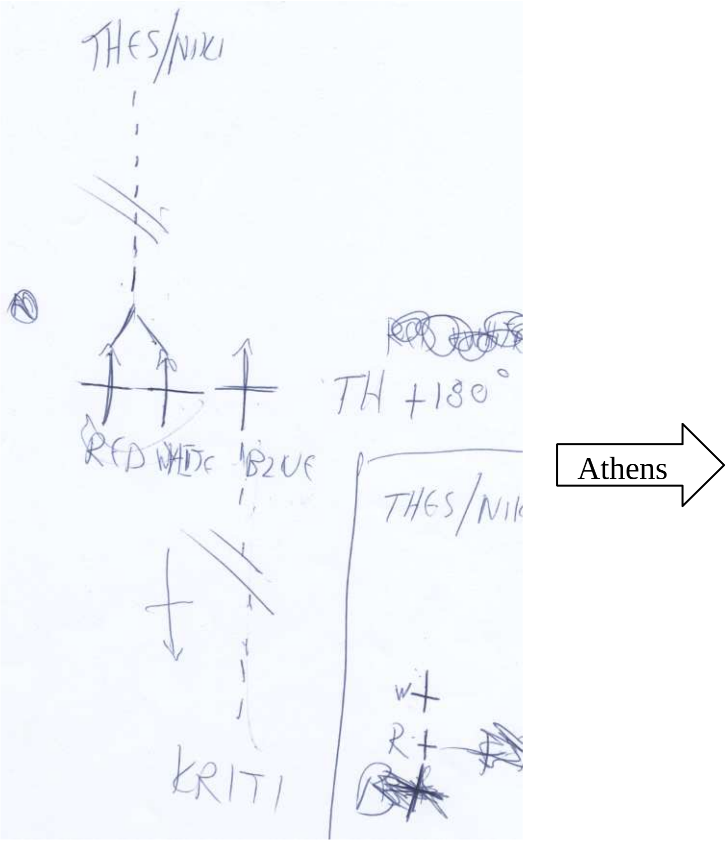
S2: *As we go up, the other, the spy, will go down contrarily, towards Crete.* […]

Let's say, if we go 10 step upwards, he goes down 10 step downwards'.

....

S1: Blue is conversely commensurate. That is to say, we go 10 meters, he goes 10 meters above. When we get to Thessaloniki, he will get to Rethymno.

From their dialogue we can assume that they were thinking about multiple displacements, as specified by the length of each displacement (i.e. 10 meters). We see that S1 seems to think of the result of these displacements as he mentions the final destination of each airplane. The interesting thing is that he argues that when the first will be at a specific city, the other will be at a specific city as well, independently of the number of displacements, implying that he used his intuition to add the vectors of displacements and find the final destination of the 2^{nd} plane.



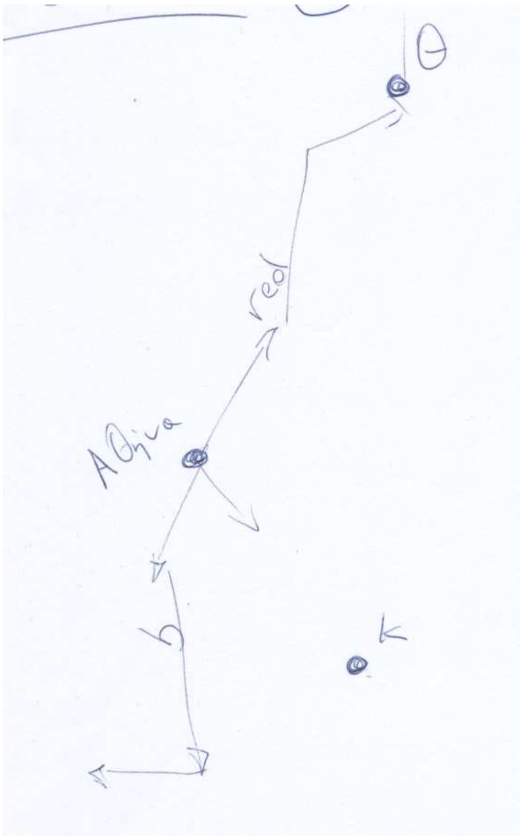


Figure 1

Figure 2

As the researcher was not sure if S1 used vectors' addition, she asked him to draw another figure and picture planes' position when the displacements would not be at the same line and asked him if the second airplane would be placed in the same city as in the first case. The student answered '*If we go to Thessaloniki, he'll be at Crete*' and draw the schema shown in figure 2. From his drawing in figure 2 we can see that although he hasn't added the vectors graphically he is thinking that the only thing that matters is the starting and the ending point. So whatever the direction of vectors would be, the second plane would be placed in a specific city, taking into account that there is a dependent relationship between the two airplane. We find this episode interesting, due to the way students use their intuitions to express mathematical meanings without using vector's terms, that is to say without mathematical formalism.

3. *Geometry*

According to the PP students were asked to construct a game for their schoolmates. In this section we'll use the activity that was included in a game created by a team. Motivated by the goal of constructing something on their own, students tried to make it as complex as possible for the other team to solve, so they thought of constructing a flight trip that had to do with mathematics. Initially they defined the route of the flight so as to form a triangle, whose vertexes would be three major cities of Greece. The goal of the game was to construct the triangle whose vertexes would be the midpoints of the first triangle's sides. The following figure shows students drawing from the activity sheet.

It was strange that students thought of an activity like that, as they were not expected to include mathematical concepts in their game. Thus, we asked them where the idea came from and S1 answered: '*We had recently written a test in geometry and I did it great, and I was thinking to use it here*.'

4. *Coordinates*

Several times, students associated the two systems of reference with the geographical information. For instance, they correlated coordinates' values with geographical borders of Greece, such as latitude and longitude in geographical coordinates.

S: Here, in position we can’t put whatever value even if we want to. Lat and long are specific.

R: Why they are specific.

S: Because we are in Greece and we can’t go all around the world. Just Greece.

R: Nice. And in height?

S: Yes. It depends.

R: It depends on what?

S: Hmm, on mountains, on each mountain’s elevation. We can’t go through mountains.

We found it really interesting that students thought geographical space as a 3D space, as they added restrictions to the third coordinate, the height.

Students also correlated spherical coordinates with the borders of Greece as shown in the following episode.

S: In r, fi and theta we have restrictions also.

R: Do we have restriction in r, fi, theta? Tell me.

S: Because we can’t go outside the map of Greece.

…… (conversation about what r, fi, theta represent)

R: Nice. And why do we have restrictions there? What is the relation between Greece and r, fi, theta?

S: We only have the map of Greece, we can’t go out of Greece.

R: What values can we use, let’s say on theta?

S: Hmm… theta and fi can take any value we want to. Just the other, r can’t be very large, because it’s how far it will go and we can’t get out of the map of Greece.

R: So, is the restriction only for r?

S: Yes.

3. SPECIFY:
   - THE KIND OF DATA YOU ANALYSED;
   - THE SPECIFIC ELEMENTS OF OBSERVATION.

3. Audio and screen recordings were analyzed verbatim in relation to students' interaction with the environment. We have focused particularly on the process by which implicit mathematical knowledge is constructed during shared student activity. As a result, in our analysis we use students' verbal transcriptions as well as their interaction with the provided representations displayed on the computer screen.
4. Students' activity sheets and notes. This would help us see the way students express their ideas in a symbolic way.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO YOUR SRQ.

IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

In designing Cruislet we wanted to integrate programming, mathematical and geographical concepts, relations and representations. New representations enabled by digital media can place spatial visualization concepts in a central role for both controlling and measuring the behaviours of objects and entities in virtual 3d environments. We have chosen the notion of vector as a mean to represent the link between 2d and 3d representations, since vectors can be considered as basic components underpinning the study of geometry and motion in space facilitating the study of 3d spatial thinking. In Cruislet, a vector-differential geometrical system co-exists with a Cartesian-geographical one in an inter-dependent way. Our analysis is mainly focused on the utilization of the different representations within open-ended exploratory tasks and the mathematical meanings that students constructed throughout this process.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

a) Characteristics of the DDA(s)

a.1 concerns about the ways mathematical objects and their interaction are represented

a.3 concerns about the ways representations can be acted on

a.5 concerns about interactions between different representation systems

a.5.1 within the DDA

These concerns guided us in identifying how students interacted with the available representations and the connection between them, to create links between mathematical and geospatial concepts. To be more specific, our analysis is mainly focused on the utilization of the different representations within open-ended exploratory tasks and the mathematical meanings that students constructed throughout this process.

b) Educational goals

b.1 epistemological concerns

b.2 semiotic concerns

Guided us in identifying issues regarding both mathematical and geographical concepts.

c) Modalities of use

c.2 concerns about the functions to be given to the DDA and their possible changes

Guided us in identifying the DDA modes of use by students according to the PP. To be more specific, our analysis is mainly focused on the mathematical meanings that students construct throughout both processes of instrumentation (carry out the activities / playing the game) and instrumentalization of the DDA (creating their own game / activity for others).

A.5.4 Analysis of ETL TE with MaLT

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The main educational goal underlying the design of our pedagogical plan with MaLT concerned the development of student's mathematical meanings for the concept of angle in 3D space when provided with challenges to construct, transform and animate 3D geometrical objects often encountered in everyday physical angle situations – such as doors, revolving doors, staircases.

Focusing primarily on students interactions with the available tools, our team aimed to study how students use the available representations in MaLT to construct meanings for the concept of angle:

- as a geometric *shape*, i.e. formed between two geometrical objects which can be segments (in 2D geometrical figures) or 2D geometrical figures (in the 3D space - dihedral angles);
- as a *dynamic amount*, indicating a change of directions both as a turn and as the result of a turn which can also be represented by a variable;
- as a *measure* represented by a number.

The above main educational goal have been articulated in the following specific educational goals in the MaLT PP:

- exploring the notion of angle as turn and measure within the 3D space (e.g. the notion of angle as a change of direction and planes in 3D space, the notion of angle between two different planes, the notion of angle between two different 3D figures)
- identifying the mathematical structure of 3D geometrical figures (e.g. distinguish the different 2D planes of the construction and relate it to the type/number of angles)
- identifying the geometrical properties of 3D objects (logical arguments to justify conclusions, relationships among angle, side lengths, perimeter areas, and volume, develop intuitions and conjectures about the geometrical properties and relations of parallelepipeds)
- understanding the relation between 2D and 3D representations when using the former to construct simulations of real 3D objects (e.g. identify the role of the 'repeat' command concerning 2D shapes in the construction of a 3D geometrical figure - between rectangle and revolving doors)
- developing fluency with the mathematical expressions to describe a 3D geometrical construction with variables
- identifying the role of variables in the construction/manipulation of 3D geometrical figures in different sizes

Remark: Though all those aspects of the concept of angle in 3D space could be singularly pursued through the planned PP, it is not reasonable to think to be able of pursuing all of them together. Actually, the choice of the specific educational goals to focus on, rests on factors like the didactic choices made by the teacher, the research focus of the researcher, the emergent perspectives during the implementation in the classroom, the progress of the activities and the available time. For example, in the implementation of the pedagogical plan in our study it emerged that dynamic manipulation provided a fruitful domain to study the construction and evolution of meanings developed by the students. Thus, we chose to exploit the respective student's activity to study in detail the role of dynamic manipulation of geometrical objects in student's conceptualisation of angle in 3D space. This choice had the effect that the role of symbolic notation in student's construction of meanings was placed in the background of the study (this is the reason for not answering the specific research question referring to the role of symbolic notation in students' construction of meanings).

Our work in designing MaLT, the respective PP and the research in our experiment is based on the following assumptions.

Assumption 1: There is a dialectic relationship between action and meaning through the mediation of the software. Computational tools provide a system through which mathematics can be expressed. Thus they orient students toward a mathematical perspective which can be traced when students use them to develop an explicit appreciation of relations (i.e. the relational invariants) and their semantics (i.e. the meanings). To achieve this tools have to illuminate structures and relationships facilitating students active engagement with particular tasks in order to make connections, develop and test hypothesis, formulate situated abstractions (explicit in the action and observable by the researcher) and communicate with their peers. All this multiplicity of roles that tools play suggests a detailed analysis of student's thinking-in-change in order to capture the subtle shifts in meaning generation and how these might have been mediated by the use of the available tools.

Assumption 2: The rich set of meanings around angle developed outside school shape students' responses in the mathematics lesson. Appropriating a mathematical perspective towards angle is not a question of replacing this set of ('informal') meanings with another ('formal'), but rather of finding ways to interrelate and connect them in meaningful ways with the relevant mathematical notions introduced in school. Because the intrinsic geometry of move in 3D space is closely related to real world experiences -such as walking or observe something flying- this kind of activities were considered to be especially efficacious in developing students' conceptualizations of angle acquired through their bodily experiences out of school. MaLT allows students to link geometrical, graphical and algebraic aspects of these experiences and connect with the notion of angle in 3D space.

Thus the achievement of the educational goals envisaged as well as the consistency between them and the hypotheses underpinning the Pedagogical Plan could be attested through the analysis of students' interactions with the available representations and the documentation of the changes in their meanings for angle in 3D space and how these might have been mediated by the use of the available tools.

Criteria for evaluating the achievement of our educational goal(s): The envisaged educational goal(s) would have been achieved if –through data analysis- we are able:

(a) to relate children's construction of meanings for the notion of angle in 3D space explicitly to their physical angle experiences;

(b) to account specifically for their difficulties in coordinating different aspects of the notion of angle as well as to throw light on the paths by which students might come to integrate their various angle concepts in 3D space;

(c) to highlight the ways by which students conceptualise angle as a spatial visualisation concept representing turn and measure through the construction and dynamic manipulation of 2D geometrical objects.

Evidence of students' achievement emerges from the analysis of video-recorded observational data, researchers' observational notes and the corpus of pupil's work on and off computer. We particularly emphasised on analysing the screen capture software files which were used to record student's voice in their groups and at the same time to capture all their actions on the screen.

It could be said that the main educational goal envisaged a-priori have been achieved. In particular, the analysis of our data brought in the foreground the following clusters of meanings constructed by pupils around the concept of angle.

- Cluster 1: Angle as a slope while navigating the turtle in 3D space

- Cluster 2: Conceptualizing a dihedral angle in 3D space

- Cluster 3: Angle as a dynamic entity for moving in different planes

According to the analysis provided in the documents concerning ReCRQ and MaLT Summary we can conclude that:

Cluster 1 is attested through criteria (a) and (b).

Cluster 2 is attested through criteria (b) and (c).

Cluster 3 is attested through criteria (a) and (c).

> (2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?
>
> EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

The hypothesis specified in our TE Portrait were the following:

Hypothesis 1: The construction of mathematical meanings as a result of students' interaction with the provided tools. This hypothesis suggests that interaction with multiple representations of geometrical objects can be a fruitful domain to challenge student's intuitions and ideas concerning spatial thinking come into play. The design of MaLT suggests that 3D geometry is a field where mathematical formalism and graphical representation of objects and relations can be dynamically joined in interesting ways and that joint symbolic and visual control may have important potential for mathematical meaning-making processes.

Hypothesis 2: The construction of meanings for 3D geometrical notions (e.g. the notion of angle) as strongly related to the connection between children's experience and mathematical structure of 3D objects. This hypothesis is based on the notion that children's conceptions of space emerge from action rather than from passive 'copying' of sensory data. The ETL team aims at connecting geometrical ideas with real tangible 3D objects that an individual experiences in everyday circumstances providing tools as a means of manipulating them and investigating their properties. An implication is that programmable geometrical constructions designed to help children abstract the notion of turtle movement in the 3D space provide a useful environment for developing their conceptualizations of 3D geometrical objects. MaLT, for example, as a Logo environment will allow learners to use their body movements to kinaesthetically pace out a geometrical construction using at the same time a mathematical language embedded in turtle's moves (consisting of its position and heading). ETL team emphasizes that students interacting with visual mathematical representations would be able to construct and deconstruct geometrical objects and develop mathematical meanings for 3D geometrical notions from that kind of process as they work collaboratively.

The possible confirmation of the above hypotheses inspiring the design of the PP and linking the use of the DDA with students' achievement is questioned through the Re-CRQ.

In synthesis, answering the question whether the interaction with multiple representations of geometrical objects could be a fruitful domain to challenge the development of student's mathematical meanings for the concept of angle in 3D space (while constructing, transforming and animating objects related to everyday physical angle situations) required to investigate the following issues related to the previously described criteria.

(a) to relate children's construction of meanings for the notion of angle in 3D space explicitly to their physical angle experiences;

With that respect, the relation that someone establishes between a representing and a corresponding represented is primarily a process of meaning-making. In other words it refers to a process of making sense of how a representing and a represented are related and if and how there is a link between a real object, its computational representation and the traditional means of representing mathematical objects in the classroom. We have had indications by the analysis that students in our study formed gradually their own 'understandings' of the essence and the functionalities of the tool and developed schemes of use which were often quite different to those intended by the designer of the computational environment and the PP.

As an indicative example we mention the episode 1 described in the Summary of our experiment. The description of the task at the introductory phase (the simulation of the take-off of an aircraft) had decisive implication on the ways by which students conceptualised angle as a slope while navigating the turtle in 3D space. In addition, the 'world' frame of reference which is inextricably linked to the body-syntonic metaphor prevalent in 2D turtle geometry contradicted with the 'vehicle' frame of reference which is by design used in turtle's navigation in the simulated 3D geometrical space of MaLT.

(b) to account specifically for students difficulties in coordinating different aspects of the notion of angle as well as to throw light on the paths by which students might come to integrate their various angle concepts in 3D space;

With that respect, the analysis indicated student's difficulties in coordinating different aspects of the notion of angle so as to integrate their various angle concepts in 3D space. As indicative examples we mention the episodes 2 and 3 described in the Summary of our experiment. Most of the students have identified dihedral angles defined by two consecutive windows (rectangles) but when attempting to describe them in mathematical terms seemed to prefer to use the terminology familiar to them from 2D geometry lessons. However, due to specific features of the visual representation of objects in the 3D scene of MaLT (i.e. the existence of vanishing points to indicate the depth of the representation) pupils were confused when attempted to identify the measure of a constructed dihedral angle in two vertical planes. Apart from the essential familiarisation with the new kinds of turtle turns (uppitch/downpitch, leftroll/rightroll) this finding could possibly be interpreted in the light of the fact that pupils who were accustomed to work with 2D representations of geometrical figures might have had difficulties in understanding the conventions used to represent a 3D object on the computer screen. However, pupils seemed to overcome such misunderstandings through the dynamic manipulation of geometrical constructions which provided them with multiple perspectives of the same 3D geometrical object, i.e. a revolving door consisted of four dihedral angles.

(c) to highlight the ways by which students conceptualise angle as a spatial visualisation concept representing turn and measure through the construction and dynamic manipulation of 2D geometrical objects.

With that respect, the analysis indicated the considerable potential of dynamic manipulation as a frame for highlighting the ways by which students might conceptualise angle as a spatial visualisation concept while constructing and manipulating 2D geometrical objects. As an indicative example we mention the finding reported in the Summary of our experiment that confirmed students' active engagement with the activity to construct the door simulation after constructing rectangles in different planes of the 3D space. The use of the two new kinds of turtle turns (rightroll/leftroll, uppitch/downpitch) coupled with pupil's experience in using variables and handling variation with 1d Variation Tool facilitated further the extension of their experimentation around the different positions of already designed 2D geometrical figures in 3D space. This kind of activity appeared to provide a fruitful domain that challenged student's intuitions and ideas about angle as a spatial quantity come into play since the use of these specific turns signalled a dynamic passage from one plane to another.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

How do student use the available representations in MaLT to construct meanings for the concept of angle in 3d space?

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

We have selected the version of Logo in MaLT as one context to explore students' ideas around the concept of angle in 3D space based on turning and directionality. After a familiarisation phase with the basic Logo commands (Introductory phase), students were engaged in building rectangles using parametric procedures in at least two different planes of the Turtle Scene (Phase 1) and experimenting with variable procedures designed to create 3D simulations like doors, revolving doors and staircases (Phase 2).

In analysing the data we were thus motivated (a) to relate children's construction of meanings for the notion of angle in 3D space explicitly to their physical angle experiences and (b) to offer a framework in which to account specifically for their difficulties in coordinating different aspects of the notion of angle as well as to throw light on the paths by which students might come to integrate their various angle concepts in 3D space. More specifically, the main focus of the study concerned the ways by which students conceptualise angle as a spatial visualisation concept representing turn and measure through the construction and dynamic manipulation of 2D geometrical objects, e.g. angle as a change of direction and simultaneously as a change of moving in different planes in 3D space, angle as a shape defined between two different planes (dihedral angle).

The analysis of our data brought in the foreground the following three clusters of meanings constructed by pupils around the concept of angle.

Cluster 1: Angle as a slope while navigating the turtle in 3D space

The move of turtle in MaLT is interrelated with the conception of angle integrating two schemes based on turning: (a) angle as a turn indicating both the act of body turning and the result of it, which inevitably involves directionality (dynamic scheme) and (b) angle as a turn represented by a number (measure scheme) (Clements et al., 1996). During the introductory phase, students were asked to navigate the turtle in the 3D geometrical space of MaLT in such a way so as to simulate the take-off of an aircraft. In this particular task pupils focused on moving the turtle around and simultaneously appeared to connect this activity with everyday experiences and physical angle contexts. From the beginning pupils experimented with all the three sets of Logo turning commands and it seemed that they had made links between the concept of angle as a turn with particular measure and that of angle as a slope. A more detailed analysis of pupil's interactions revealed that students oscillated between two different frames of reference: (a) world frame: defined in terms of directions 'up' and 'down' and (b) a vehicle frame: typically associated with the orientation of a moving entity, here the turtle. Though at the initial position of the turtle the 'vehicle' frame of reference coincides with the 'world' frame of reference the use of roll turns might result to contradict one another. Thus we can argue that although 3D simulated space is closer to real life and every-day experiences, the body-syntonic metaphor appears to be less strong in 3D turtle geometry than in 2D. For instance, we can easily simulate 2D turtle motion with our body but we cannot simulate 3D turtle's motion. Thus, it seems that the body-syntonic frame, which is inextricably linked with the 'world' frame in real 3D space, should be shrunk in favour of the 'vehicle frame' underlying the turtle move in the simulated 3D space.

In the language of DF, the navigation of the turtle by the students in 3D space provided the context in which to analyse the role of concerns as tool ergonomy, characteristics of the implementation of mathematical objects and the possible actions on these objects to characteristics of the possible interaction between students and mathematical knowledge around the concept of angle.

Cluster 2: Recognizing (or conceptualizing) a dihedral angle in 3D space

A second cluster of meanings concerned the conceptualisation of a dihedral angle in 3D space. This kind of activity appeared in Phase 1 of experimentation when the teacher/researcher asked pupils to construct rectangles using parametric procedures in at least two different planes of the Turtle Scene simulating the construction of windows in a virtual room. The need to design figures in different planes of the 3D space challenged pupils to move the focus of their attention from directed turns between lines and planes to directed turns between two similar geometrical figures. Most of the students have identified dihedral angles defined by two consecutive windows (rectangles) but when attempting to describe them in mathematical terms seemed to prefer to use the terminology familiar to them from 2D geometry lessons. However, due to specific features of the visual representation of objects in the 3D scene of MaLT (i.e. the existence of vanishing points to indicate the depth of the representation) pupils were confused when attempted to identify the measure of a constructed dihedral angle in two vertical planes. Apart from the essential familiarisation with the new kinds of turtle turns (uppitch/downpitch, leftroll/rightroll) this interpretation could possibly be interpreted in the light of the fact that pupils who were accustomed to work with 2D representations of geometrical figures might have had difficulties in understanding the conventions used to represent a 3D object on the computer screen. However, pupils seemed to overcome such misunderstandings through the dynamic manipulation of geometrical constructions which provided them with multiple perspectives of the same 3D geometrical object, i.e. a revolving door consisted of four dihedral angles. The more the students appeared accustomed to the conventions used in the 3D simulated space the more they were able to coordinate the visual characteristics of the dihedral angles with their measure related to the turtle's turns from one plane to another.

In the language of DF, student's difficulties to distinguish the geometrical characteristics of dihedral angles as visualised in the Turtle Scene bring in the foreground the importance of specific tool characteristics and particularly the possible conflict between the ways by which the mathematical knowledge of a specific domain is implemented in a DDA and the forms of didactic interaction provided by the DDA. The problem created by a specific representation of MaLT (way of visual perspective) seemed to have been resolved for the students due to the joint use of visual, symbolic and dynamic manipulation representational registers. These tools provided the means that enabled to recognise the mathematical features of dihedral angles though experimentation, observation of different perspectives and dynamic manipulation. So we can conclude that both the implementation of the knowledge of the domain and the didactic interaction can be approached through different perspectives, which are neither independent nor mutually exclusive.

Cluster 3: Angle as a dynamic entity for moving in different planes

A third cluster of meanings in our data analysis concerns the concept of angle as a dynamic entity for moving in different planes. Initially students have focused on changing planes as a result of changing the turtle's position. The use of the two new kinds of turtle turns (rightroll/leftroll, uppitch/downpitch) coupled with pupil's experience in using variables and handling variation with 1d Variation Tool facilitated further the extension of their experimentation around the different positions of already designed 2d geometrical figures in 3D space. This kind of activity appeared to provide a fruitful domain that challenged student's intuitions and ideas about angle as a spatial quantity come into play since the use of these specific turns signalled a dynamic passage from one plane to another. For instance, most of the groups of pupils found engaging the activity to construct the door simulation after constructing rectangles in different planes of the 3D space.

In the language of DF, student's active engagement to construct or to experiment with simulations of concrete objects that involve 'continuous' turning in the space seems to be related with the strong links between tool characteristics and educational goals with the given tasks. In this case we challenged pupils to experiment with these type of simulations having an epistemological consideration in mind: to provide a basis for pupils intuitions come into play through the use of the rotation commands. As far as they have incorporated the use of these commands in their activities pupils were able to coordinate the interplay between aspects of angle as dynamic entity for moving in different planes by simulating 3D objects. In these cases, the mathematisation of pupils' responses while experimenting with such simulations was inextricably related to the kinesthetic nature of the computer feedback translated in the context of the given activities. As far as the DF is concerned this point is useful in considering mathematics as a domain of knowledge and as a field of practice emerging in the context of specific educational goals interrelated with specific activities.

3. SPECIFY:

- THE KIND OF DATA YOU ANALYSED;

- THE SPECIFIC ELEMENTS OF OBSERVATION.

MaLT was exploited as a 'tool to think with' and as a field of experimentation and exploration rather than as a means of calculating correct answers. The classroom activities described in the MaLT pedagogical plan have been perceived as innovative for the actors involved since they consisted of small group project work based on the use of exploratory software and open-ended tasks allowing multiple explorations and personal forms of reasoning. In order to describe pupils' learning trajectories as they happen in real time the

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The elements of observation thus can be divided in four groups:

(a) pupil's interaction with the available tools

(b) pupil's communication within their groups

(c) teacher's interventions

(d) pupil's non-verbal modes of interaction (e.g. gestures, facial expressions)

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ.

   IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

The main theoretical frames and constructs adopted in the present study include *constructionism, social constructivism, situated abstraction, conceptual field* and *instrumental genesis*. Based on these theoretical origins we draw on the idea of teaching and learning mathematics with the use of technology with learners as central sense-making agents while interacting with specially designed exploratory computational tools and representations viewed as integral to mathematical activity rather than an external aid to internal cognitive processes. The constructionist/social-constructivist framework expects students to interact with and manipulate the representations provided by the tool, making sense of their behaviours through this interaction with the computer environment and with the social context of the classroom. The constructionist theoretical perspective of the MaLT pedagogical plan was based on the assumption that programmable geometrical constructions designed to help children abstract the notion of turtle movement in the 3D space provide a useful environment for developing their conceptualizations of geometrical objects, like angles.

The wording of the reformulation of the CRQ by our team specified the priority of the student’s engagement in experimenting with the available tools by introducing a distinction between the term ‘representations’ (which exists in the CRQ) with the phrase “student’s use of representations”. The theory of constructionism –and specifically the theoretical construct of *situated abstraction*- influenced our decision to replace the phrase referring to the relation between the representations and the user (“be put in relationship”) with the phrase “to construct meanings”. This theoretical perspective indicates that the relation which someone establishes between a *representing* and a corresponding *represented* –in the terms of Minimal Theoretical Framework- is conceived as a process of making sense how they are related and if and how they realize a link between a mathematical object, its technological representation and its relation to the traditional means of representation in the classroom. The term *meanings* suggests that within the activities students were expected to experiment with different strategies and, more importantly, attach personal meanings to the results of their activities shaped by the artefact in ways that lead them to diverge from curriculum mathematics.

For the analysis we transcribed verbatim the audio recordings of three groups of students throughout the teaching sequence (using the HyperCam2 files coupled with occasional video recordings taken from a camera) and we also selected significant learning incidents from the work of all groups in the classroom. In analyzing the data we firstly looked for instances where meanings related to the visualisation and conceptualisation of the notion of angle in the simulated 3d geometrical space were expressed by the students. Our objective was to gain insight into:

(a) the nature of the mathematical meanings constructed by pupils around the conceptual filed of angle

(b) the ways in which meaning generation interacted with the use of the available tools.

The unit of analysis was the episode, defined as an extract of actions and interactions developed in a continuous period of time around a particular issue. In most cases word episodes were meaningless if they were not related to the sequences of actions that students carried out while working on the computer or more importantly if they were not related to the gestures they used and their body movement. In these cases we based our analysis on the joint study of the transcribed interaction between the actors participating in a specific episode with the available video recordings. The episodes were selected:

(a) to have particular and characteristic bearing on the pupil’s interaction with the available tools accompanied with the constructed mathematical meanings;

(b) to represent clearly the kind of activity that was going on.

We used these as the main means of presenting and discussing the data.

The construct of situated abstraction was central in this process. An important corollary of this is that we maintain a predilection for studying the potentials of alternative representations which afford the learner the opportunity to move smoothly between different meanings derived from language and actions and simultaneously to build new meanings. For instance let us walk you through a specific episode concluded in the summary of the ETL experiment. In episode 2 we initially based our analysis to the ways by which pupils' intuitions of moving in 3D space were coordinated (or not) with to the available visual representations in MaLT. This way we were able to define student's difficulties in describing the mathematical features of the dihedral angles that they had constructed.

However, in the next phases of the analysis we were interested to see if and how pupils seemed to overcome these difficulties and especially to capture the role of tools and task in this process, i.e. we were interested to study the ways in which the students interacted with the available representations and the ways in which the meanings they constructed structured and were structured by them. So, in the next part of the analysis we were able to account for the role of the joint use of visual, symbolic and dynamic manipulation representational registers in providing the basis for the pupils to go beyond the simple visual recognition of angular relationships in 3D space to their expression and further elaboration. In this process the instrumental issue was taken into consideration in conjuction with constructionist activity giving rise to a dialectic by which learner and artefact are mutually shaped in action. This emerging dialectic between learners and instruments seemed to offer a framework in which to account specifically for meaning-making processes concerning angular relationships in the 3d space.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

With respect to the characteristics of the DDA our analysis was guided by the following top research concerns:

- concerns about the ways mathematical objects and their interaction are represented
- concerns about interactions between different representation systems within the DDA

With respect to the educational goals our analysis was guided by the following top research concerns:

- Epistemological concerns
- Semiotic concerns

With respect to modalities of use our analysis was guided by the following top research concerns:

- concerns about the functions to be given to the DDA
- concerns about the relationship between knowledge referred to the DDA functioning and knowledge referred to the educational goals

Throughout the analysis of our data we tried to have all the above concerns in mind and to carry out a multi-level analysis. Although the clusters of analysis were defined in relation to the different aspects of the notion of angle and the evolutive character of relative meaning generation, the analysis of each cluster synthesizes all the above concerns.

For instance, in cluster 1 it is evident that the construction of meanings in relation to the concept of angle as a slope is closely related to the modalities of use: to the functions given to the DDA in relation to the knowledge referred to the educational goals and tasks (the simulation of the take-off an aircraft). In cluster 2, the way students conceptualised dihedral angles is interweaved with the characteristics of the DDA and the way mathematical objects are represented as well as to the way different representational systems are interconnected within the DDA (e.g. visual representation combined with symbolic code and dynamic manipulation of geometrical objects). As students got more and more capable of handling and synthesizing different aspects of the notion angle in cluster 3, it is more eloquent how the educational goals of ETL's PP and the activities carried out are interacting with the characteristics of the DDA and the modalities of use shaping students' learning trajectories. In this part of the analysis we took also into account the ways by which semiotic concerns provided by the DDA were interrelated in student's experimentation with epistemological concerns underlying the nature of meanings for angle in 3D space constructed by the pupils and the relation of this kind of meanings with the official knowledge concerning angle in 3D space (i.e. students seemed to conceptualise angle in innovative ways embedded in different physical angle situations without reference to formal definitions as usually happens in the respective curricular activities).

Specific Research Questions

This section is meant to collect your answers to your SRQ, contained in the TE Portrait.

As one can notice, we are proposing the (more or less) same frame to articulate both the answers to the ReCRQ and those to the SRQs.

For each possible SRQ fill the following.

1. REPORT YOUR SRQ.

How do students use the dynamic manipulation tools available in MaLT to construct meanings for the concept of angle in 3D space?

2. ANSWER YOUR SRQ. WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

We have selected the version of Logo in MaLT as one context to explore students' ideas around the concept of angle in 3D space based on turning and directionality. After a familiarisation phase with the basic Logo commands (Introductory phase), students were engaged in building rectangles using parametric procedures in at least two different planes of the Turtle Scene (Phase 1) and experimenting with variable procedures designed to create 3D simulations like doors, revolving doors and staircases (Phase 2).Throughout the implementation of ETL's pedagogic plan 13-year-old pupils were engaged in exploring the mathematical nature of angles while controlling and measuring the behaviours of geometrical objects in the simulated 3D space of MaLT. The move of turtle in MaLT is interrelated with the conception of angle integrating two schemes based on turning:

(a) angle as a turn indicating both the act of body turning and the result of it, which inevitably involves directionality (dynamic scheme) and

(b) angle as a turn represented by a number (measure scheme) (Clements et al., 1996).

In particular we were interested in the way mathematical ideas are constructed gradually by the students while drawing upon the functionalities and phenomenological cues available in MaLT. One of the key functionalities in MaLT is dynamic manipulation. Geometrical constructions can be expressed with the use of variables and dynamically manipulated by specially designed computational tools called variation tools. Students were able to dynamically manipulate conventional 2D and 3D representations of geometrical figures by using these specially designed variation tools rendering parametric procedures descriptors of evolving geometrical objects in relation to the value of a variable or of a set of variables. An as result students had the chance to observe the behaviour of the varying parts in relation to each other and to the invariant ones and to acquire a sense of generality and abstraction underlying some static instances of the mathematical structures.

Drawing upon the preliminary analysis of our results it seems that the dynamic manipulation metaphor available in MaLT offered a framework in which to explore angle as a dynamic amount in various cases where angle was considered:

- As a decisive element of turtle's position and orientation in 3D space
- As a constitutive element of geometrical figures

- As a means of specifying the position of geometrical figures in space and in relation to each other.

In particular the analysis of our data brought in the foreground the following two clusters of meanings constructed by pupils around the concept of angle strongly influenced by the use of dynamic manipulation tools available.

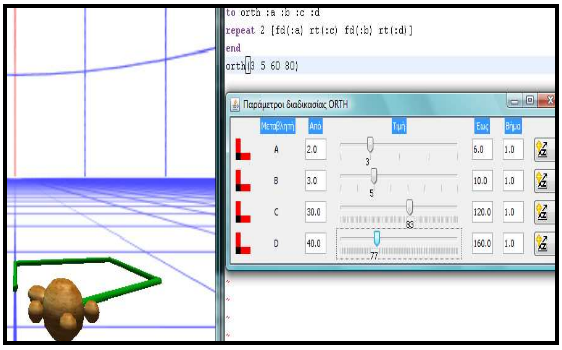
Cluster 1: Angle as a dynamic amount for the construction of 2D figures

In the following episode students are trying to construct a rectangle while dynamically manipulating the four variables of a procedure that was given to them and creates a crooked line. Students decided to use the 1d Variation Tool in their experimentation so as to change the values of the respective variables and thus have immediately the graphical outcome visualised on the screen. Thus, it seems that the dragging modalities of the software facilitated experimentation and provided a link between mathematical variation and its geometrical representation.

Episode 1

E *So, can you find a way to close this figure? What can we do to try to make this line a rectangle?*

S2 *We can make it go forward and then this way changing the values here (he means in the Logo Editor Componnet)that says orth (3 5 60 80). We must think how. As I don't know, I will make some trials.*



S1 *But you don't have to make so many trial. It is boring. We can make these trials with that that you can minimize and maximise.*

He means the 1d variation tool

s2 *We can make it smaller or bigger, don't we? We will work in this window and we will change it continually. You are right in fact we can make trials and see the values moving on the screen!*

M1 *That's rightς*

Handling the variation of the variable values for turns the notion of angle came in the foreground both as a constitutive element of turtle's position and orientation and as a constitutive element of 2D geometrical figures. Additionally, students seemed to approach angle not only as a turn with a fixed measure, but also as a turn with a dynamic amount, in other words as turn with a measure that can by dynamically handled and changed sequentially using the functionalities of 1d Variation Tool. The various instances of the figure available offered students the chance to check their intuitions and conjectures and facilitated the formulation of conclusions concerning the relationships between the angles of a rectangle.

Cluster 2: Angle as a dynamic entity for moving in different planes

A second very interesting cluster in our analysis concerns the concept of angle an a dynamic entity for moving in different planes. The use of the two new kinds of turtle turns (rightroll/leftroll, uppitch/downpitch) coupled with pupil's experience in using variables and handling variation with 1d Variation Tool facilitated further the extension of their experimentation around the different positions of already designed 2D geometrical figures in 3D space. This kind of activity appeared to provide a fruitful domain that challenged student's intuitions and ideas about angle as a spatial quantity come into play since the use of these specific turns signalled a dynamic passage from one plane to another. For instance, most of the groups of pupils found engaging the activity to construct the door simulation after constructing rectangles in different planes of the 3D space.

In many cases where they were developing procedures students decided to use not a fixed turn measure but a variable. In the following episode students decide to use a variable so as to progressively move the door that they have created in the horizontal plane to the vertical one. It seems that the use of the variable gave a more realistic effect in their construction in addition to the chance of easily experimenting with its measure.

Episode 2

S1	*Lets do up*	
S2	*a*	
S1	*No, 90;*	
S2	*No! a*	
S1	*Up [she moves her hand like moving a door]*	
S2	*Up…the whooole. So, what I need?*	
S1	*a.*	
S2	*So, we will slowly create a door.*	*She shows with her hand a progressive movement of the rectangle between the horizontal and the vertical plane.*
S1	*up(:a) and now…*	

S2 *Now stop. We did up to create the angle, then forward, then right so now we need rt(:d) and then forward.*

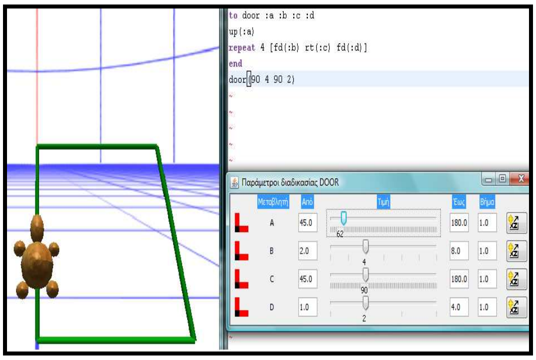
to door a b c d

up(:a)

repeat 4 [fd(:b) rt(:c) fd(:b)]

end

door(90 4 90 2)



Students progressively got more and more capable of handling different aspects of angle simultaneously. For instance in the following episode students are experimenting using the 1d variation tool with the variables of the procedure 'Slide' (which was given ready-made to them) so as to create a sliding door moving around. It seems that students create meanings in relation to angle as:

- a constitutive element of a figure which is defined and stay fixed (variable c)
- as a means to move from the horizontal plane to the vertical one in relation to the viewing axis of the user which is again defined and stay fixed (variable d)
- as a means of constantly changing levels around x axis (variable e)

Episode 3

S2 *Ooh! What is this!!!Wait!*

They run the code

to slide :a :b :c :d :e

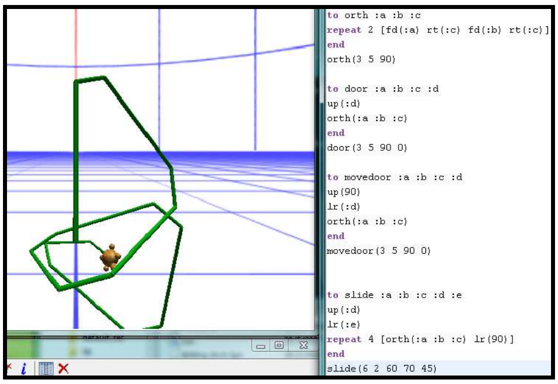
up(:d)

lr(:e)

repeat 4 [orth(:a :b :c) lr(90)]

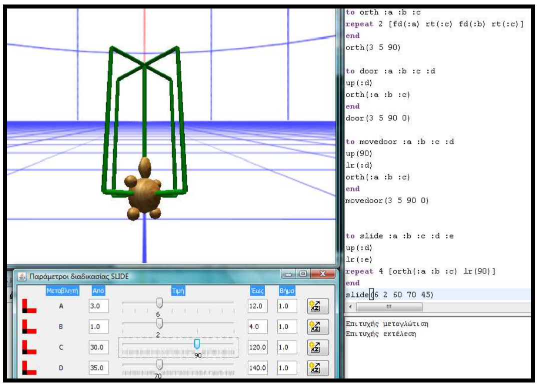
end

slide (6 2 60 70 45)



We should move it here first, it's the angle of the rectangle (points at it and moves the variable c) so as to become like this (means the door) and then probably here so that it turns like this (points and moves the variable e). Let's see…

S1	*Yes, it definitely turns around with this as it has lr.*	*They activate the one-dimensional variation tool.*
S2	*Yes, but we don't only want it to turn, we also want it to move even further down.*	
M1	*I should change here (He shows the final value of the variable d. He puts a final value d=90).*	



M2	*Yes, 90 is fine.*	
M1	*Now with this (pointing the variable e) it turns around normally (points at the angles between the levels).*	
M2	*Fine!!! Perfect [Moves with 1d variation tool the variable e].*	

In the course of their experimentation and while changing dynamically the values of the variables of the procedure 'slide', students were able to recognise the four consecutive right dihedral angles created between the four rectangles around X axis. However it should be stressed that the simulation of the motion of rectangles (that represented a sliding door) around X axis as a result of the use of 1d Variation Tool, gave students the chance to see the dihedral angles created from different perspectives. Viewing dihedral angles from different perspectives minimized the 'distorting' effects of 3D representations' conventions used in MaLT scene that had mislead students in other static 3D constructions in previous phases of the experimentation. Dynamic manipulation prompted students to focus more on the measure of turtle's turn in Logo code in order to decide about the kind of dihedral angle represented.

Finally, we quote another episode which highlights the way students approached different aspects of the concept of angle while experimenting with a procedure that was given to them and had as a result the simulation of the opening and closing of the pages of a book. In this experimentation, through the use of 1d Variation Tool, angle was approach as a turn with a dynamic measure:

o while changing sequentially the planes around X axis clockwise or counterclockwise through the dynamic manipulation of the variable b that defines turtle's position in 3D space

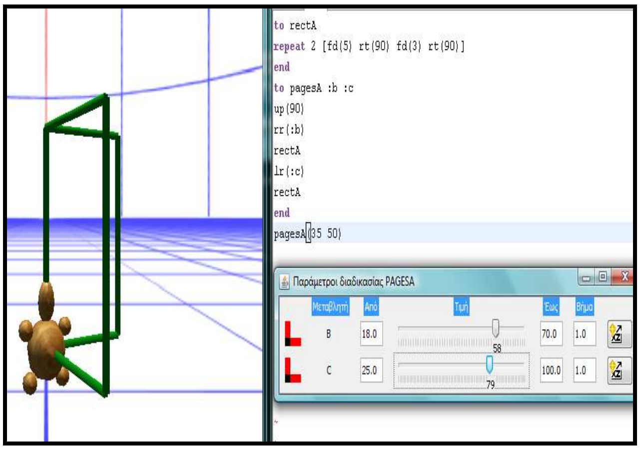
o while changing sequentially the measure of the dihedral angle created between the two rectangles that represent the two pages of a book.

Episode 4

It's like that.

Now we have to see how they move

She activates the 1d variation tool and moves firstly the slider of variable b and then the slider of variable c.



to rectA

repeat 2 [fd(5) rt(90) fd(3) rt(90)]

end

to pagesA :b :c

up(90)

rr(:b)

rectA

lr(:c)

rectA

end

pagesA(35 50)

What do you observe? Tell me.

Mμ, It should open because in the procedure there is lr andı rr

Tell me what happens when you move the 1d Variation tool?;

When you change this, the turtle turns, thus this angle

He means the angle between

does not change at all.	*the two planes.*
Yes	
While here the one side turns and the turn changes and thus the whole angle changes.	*He changes with the slider the value of variable c*
So when you change this what does it happens?	*She means the slider f variable b.*
With b the turtle's turn changes, it turns and makes a whole r ound as if it carries both pages	
Yes.	
Now with c…it turns and only the one side closes	*He means the one plane of the dihedral angle.*

It seems that the use of variation tools facilitated not only dihedral angles' recognition but also its approach as an angle with a dynamic measure that can be handled and described drawing upon everyday experience (simulating the opening and closing of books' pages), body movement (opening and closing hands) and mathematical terms used in 2D geometry lessons.

In the language of DF, the dragging modalities of the software and the way geometrical representations could be acted upon facilitated experimentation and brought in the foreground issues related to the way different representational systems, (e.g. geometrical representation and algebraic notation) are interrelated. This interplay between the different representational systems offered multiple entries to the mathematical objects represented and was conducive to the construction of shared meanings between the perceivable representing and the corresponding represented. Student's active engagement to construct or to experiment with simulations of concrete objects that involve 'continuous' turning in the space seems to be related with the strong links between tool characteristics and educational goals within the given tasks. In this case we challenged pupils to experiment with such kind of simulations having an epistemological consideration in mind: to provide a basis for pupils intuitions come into play through the use of the rotation commands. As far as they have incorporated the use of these commands in their activities pupils were able to coordinate the interplay between aspects of angle as dynamic entity for moving in different planes by simulating 3D objects. In these cases, the mathematisation of pupils' responses while experimenting with such simulations was inextricably related to the kinesthetic nature of the computer feedback translated in the context of the given activities. As far as the DF is concerned this point is useful in considering mathematics as a domain of knowledge and as a field of practice emerging in the context of specific educational goals interrelated with specific activities.

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   IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

The main theoretical frames and constructs adopted in the present study include *constructionism, social constructivism, situated abstraction, conceptual field* and *instrumental genesis*. Based on these theoretical origins we draw on the idea of teaching and learning mathematics with the use of technology with learners as central sense-making agents while interacting with specially designed exploratory computational tools and representations viewed as integral to mathematical activity rather than an external aid to internal cognitive processes. The constructionist/social-constructivist framework expects students to interact with and manipulate the representations provided by the tool, making sense of their behaviours through this interaction with the computer environment and with the social context of the classroom. The constructionist theoretical perspective of the MaLT pedagogical plan was based on the assumption that programmable geometrical constructions designed to help children abstract the notion of turtle movement in the 3D space provide a useful environment for developing their conceptualizations of geometrical objects, like angles.

The wording used in the above formulated SRQ specifies the priority given by our team on student's engagement in experimenting with the available tools indicating that the relation which someone establishes between a *representing* and a corresponding *represented* –in the terms of Minimal Theoretical Framework- is conceived as a process of making sense of how they are related as well as a process of identifying and establishing link between a mathematical object, its technological representation and its relation to the traditional means of representation in the classroom. The theory of constructionism –and specifically the theoretical construct of *situated abstraction*- influenced our decision to use the term *meanings* suggesting that within the activities students were expected to experiment with different strategies and, more importantly, attach personal meanings to the results of their activities shaped by the artefact in ways that lead them to diverge from curriculum mathematics.

For the analysis we transcribed verbatim the audio recordings of three groups of students throughout the teaching sequence (using the HyperCam2 files coupled with occasional video recordings taken from a camera) and we also selected significant learning incidents from the work of all groups in the classroom. In analyzing the data we firstly looked for instances where meanings related to the visualisation and conceptualisation of the notion of angle in the simulated 3D geometrical space were expressed by the students. Our objective was to gain insight into:

(a) the nature of the mathematical meanings constructed by pupils around the conceptual filed of angle

(b) the ways in which meaning generation interacted with specific tools functionalities and in particular with the use of Variation Tools.

The unit of analysis was the episode, defined as an extract of actions and interactions developed in a continuous period of time around a particular issue. In most cases word episodes were meaningless if they were not related to the sequences of actions that students carried out while working on the computer or more importantly if they were not related to the gestures they used and their body movement. In these cases we based our analysis on the joint study of the transcribed interaction between the actors participating in a specific episode with the available video recordings. The episodes were selected:

(a) to have particular and characteristic bearing on the pupil's interaction with the available tools accompanied with the constructed mathematical meanings;

(b) to represent clearly the kind of activity that was going on.

We used these as the main means of presenting and discussing the data.

The construct of situated abstraction was central in this process. An important corollary of this is that we maintain a predilection for studying the potentials of alternative representations which afford the learner the opportunity to move smoothly between different meanings derived from language and actions and simultaneously to build new meanings. We were interested in identifying students difficulties and seeing if and how pupils seemed to overcome these difficulties and especially to capture the role of tools and task in this process, i.e. we were interested to study the ways in which the students interacted with the available representations and the ways in which the meanings they constructed structured and were structured by them. For instance, episodes 2 and 3 in our analysis shows how students progressively constructed meanings in relation to the dihedral angles that they had created and handled dynamically with the 1d Variation Tool. It seems that progressively the use of variation tools facilitated not only dihedral angles' recognition but also its approach as an angle with a dynamic measure that can be handled and described drawing upon everyday experience (simulating the opening and closing of books' pages), body movement (opening and closing hands) and mathematical terms used in 2D geometry lessons.

In the our analysis we were able to account for the role of the joint use of visual, symbolic and dynamic manipulation representational registers in providing the basis for the pupils to go beyond the simple visual recognition of angular relationships in 3D space to their expression and further elaboration. In this process the instrumental issue came in the foreground in conjuction with constructionist activity giving rise to a dialectic by which learner and artefact are mutually shaped in action. This emerging dialectic between learners and instruments seemed to offer a framework in which to account specifically for meaning-making processes concerning angular relationships in the 3D space.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

With respect to the characteristics of the DDA our analysis was guided by the following top research concerns:

- concerns about the way representations can be acted on
- concerns about interactions between different representation systems within the DDA

With respect to the educational goals our analysis was guided by the following top research concerns:

- Semiotic concerns

With respect to modalities of use our analysis was guided by the following top research concerns:

- concerns about semiotic issues

Throughout the analysis of our data we tried to have all the above concerns in mind and to carry out a multi-level analysis. Although the clusters of analysis were defined in relation to the different aspects of the notion of angle and the evolutive character of relative meaning generation while using the 1d Variation Tool, the analysis of each cluster synthesizes all the above concerns. For instance in cluster 1 it is evident that the construction of meanings in relation to the concept of angle as a dynamic amount while constructing 2D figures is closely related to the dragging modalities of MaLT and the number-line metaphor used which facilitated experimentation providing a link between geometrical figures and symbolic notation. Students experimented with the value of variables in ready-made procedures in order to find the right-(fixed) measure of angle that would create the figure that they had in mind.

In cluster 2 the way students conceptualised dihedral angles is interweaved with the characteristics of the DDA and the way mathematical objects are represented and can be acted upon as well as to the way different representational systems are interconnected. Students progressively got more and more capable of handling and synthesizing different aspects of the notion of angle, while coordinating the use of variables and other symbolic notation with its geometrical counterparts. For instance students used variables in their procedures in combination with the dynamic manipulation functionalities of the DDA in order to simulate the progressive change of planes in the simulated 3D space as well as in order to animate 3D figures. Thus, it could be pointed out that the semiotic activity observed came as a result of the interplay of didactical functionalities.

6. Is your SRQ meant to contribute to provide an answer to your RE-CRQ? If yes, how?

Students' learning trajectories as well as the multifaceted and multileveled process of meaning construction cannot be segmented in mutually exclusive or non interconnected research questions. However we consider that the more broadly formulated RE-CRQ can be enriched by a number of SRQ that would shed light in its various aspects. In this context the SRQ that we tried to answer above examines more thoroughly one aspect of the RE-CRQ, that of DDA's dynamic manipulation functionalities and the way it affected meaning construction in relation to the concept of angle.

Note

During the implementation of the pedagogical plan in our study it emerged that dynamic manipulation provided a fruitful domain to study the construction and evolution of meanings developed by the students. Thus, we chose to exploit the respective student's activity to study in detail the role of dynamic manipulation of geometrical objects in student's conceptualisation of angle in 3D space. This choice had the effect that the role of symbolic notation in student's construction of meanings was placed in the background of the study. This is the reason for not answering the second specific research question referring to the role of symbolic notation in students' construction of meanings contained in the Teaching_Experiment_Analysis_Guidelines.

A.5.5 Analysis of ETL TE with MoPiX

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The main educational goal addressed through the design of the MoPiX Pedagogical Plan, as it was stated in the Teaching Experiment Guidelines (July 2007), concerned the students' construction of mathematical meanings regarding the role of the MoPiX algebraic equations and the relationships between them, while representing phenomena such as collisions and motions and experimenting (e.g constructing, deconstructing and reconstructing) with the corresponding animated models.

The meanings that the students would construct about the role of the equations were not explicitly defined during the development of the Pedagogical Plan. We expected those meanings to emerge as the students would interact with the MoPiX DDA and engage in collaborative activities which were challenging by design (e.g changing a half-baked microworld). Thus, the main educational goal addressed a the Pedagogical Plan was deliberately not correlated to a specific mathematical concept or domain, as it would be for example the concept of function or the notion of the variable.

This design choice is also reflected to the specific educational goals stated in each phase of the Pedagogical Plan which are related to the activities themselves and to the characteristics of the DDA. According to those goals we expected students to:

1. Observe and analyse the objects' animated behaviours and the properties
2. Modify the objects' behaviour and the properties by adding or removing equations
3. Edit one-object or multi-object already existing equations in order to describe properties and behaviours
4. Construct new one-object or multi-object equations in order to describe new properties and behaviours
5. Connect the visual representation (animation) of the objects' behaviour to the symbolic representation (equations) of the behaviours
6. Collaborate in pairs discussing, forming and testing hypotheses, negotiating

and reaching in joint conclusions concerning the object' s behaviour after adding/removing equations to/from the object

The achievement of the main educational goal will be attested through the achievement of the specific educational goals as they are mentioned above.

In order to attest the achievement of our educational goal, we formed a Research Question that would require the analysis of the data to be conducted in such a way that would bring into the surface several incidents in which students did construct mathematical meanings about the role of the equations while engaging in the PP's activities and accomplishing the aforementioned specific goals.

The Research Question concerns the ways in which students constructed meanings about the role of the equations and the analysis performed with regard to this RQ indicates the existence of numerous episodes in which at least one of the specific goals is achieved. We classified them in three categories:

- Construction of meanings about the role of an equation through the interpretation of its symbols.
- Construction of meanings about the role of an equation through the editing of its symbols.
- Construction of meanings about the role of an equation through its conceptualization and development.

As it becomes apparent, the achievement of the main educational goal can be attested though the analysis conducted with respect to our Research Question, as it is presented in the "Common Research Question" section.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONSHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

The hypotheses made when designing the PP that link the use of the DDA to the envisaged educational goal as they were stated in the TE guidelines:

The activities designed for the specific part of the Pedagogical Plan where the selected educational goal is addressed aim to introduce students into the "Juggler" half – baked microworld and its functionalities. This microworld consists of several objects ("hands" and "balls") whose behaviours are linked to each other's. The connection between the objects is perceived both by means of symbolic and visual representations.

Students are expected to explore the functionalities of the microworld and deconstruct the "Juggler" existing model so as to define the equations underpinning the behaviour and properties of each object both as a standalone object and with regard to the others. This means that students will use the "Flip Object" feature so as to make visible the equations assigned to each object, remove or add equations in order to alter the objects' behaviour and properties according to their understanding and execute the model so as to receive a visual feedback.

The deconstruction procedure will afford them with the opportunity to gain understanding of the mathematical structure of the equations used in MoPiX environment to express behaviours and properties.

1st Remark:

The hypotheses as they were presented in the TE Guidelines (July 2007) indicate that the main educational goal (i.e the students' construction of meanings about the role of the equations) was addressed only for a specific phase of the Pedagogical Plan, the phase in which the students changed the "Juggler" half-baked microworld. Having in mind that -in this phase- the students would engage in activities that were designed to be particularly challenging for them (i.e play with the "Juggler" according to the existing rules, deconstruct the model underpinning its behaviour and modify it so as to express their personal ideas), we focused the wording of the hypotheses almost exclusively to this phase of the experimentations.

However, the construction of meanings about the role of the equations was expected to also emerge during the first phase of the experimentations. For this phase, we had designed activities during which the students would be invited to deconstruct a model consisting of a single object (i.e the "One red ball") so as to determine the object's behaviour and create a second object that would have the exact same behaviour as the first one.

In this process we expected students to assign and remove equations from the objects, observe and discuss with peers the animation generated and make connections between the visual and the symbolic representation system. Since the second object to be constructed required the editing of the already existing equations, we also expected students to start modifying equations and through this process start constructing meanings about the role of the equations.

2nd Remark:

Moreover, what we omitted in the TE Guidelines (July 2007) is to refer to the construction of equations (apart from the editing) and formulate the corresponding hypothesis.

The activities we have designed for the PP and the microworlds themselves invite students to change the microworlds' functionalities so as to create a unique artefact, possibly distinctly different to the original one. In order to do so the students are expected to modify the equations comprising the microworld's model and possibly invent new symbols and equations that will describe the new behaviours they would wish to assign to their objects.

By reformulating the hypotheses made when designing the PP –not after receiving feedback from the experimentation and the analysis process, but after reconsidering the wording of the hypotheses as they were presented in the TE Guidelines (July 2007)- we try to make explicit how the use of the DDA is linked to envisaged educational goal. Each of the hypotheses incorporates in its wording the educational goal and thus confirmation of the hypotheses contributes to the attestation of the main educational goal.

Hypothesis 1:

The students would construct meanings about the role of the equations as they connect the symbolic representation to the graphical representation available in MoPiX.

The activities designed for the PP and the MoPiX environment itself provide students the opportunity to modify the model underpinning the behaviour of the objects present on the Stage by adding or removing equations. The equations added/removed could be equations selected from the "Equations Library" or equations that the students developed themselves by editing an existing equation or by constructing a new one.

The equations assigned to the objects form a model that the students may execute so as to generate its visual representation. Linking the visual representation (i.e the animation generated) to the symbolic representation (i.e the equations assigned to the objects) will afford students the opportunity to attribute meaning to an equation -by referring to its symbols or its structure- and thus construct meanings about the role of this equation.

Hypothesis 2:

The students would construct meanings about the role of the equations by editing already existing equations or by constructing new ones.

The PP we developed incorporates the use of a specific kind of microworld, the “Juggler” half baked microworld. This microworld -by its own nature- and the activities we have designed for the PP, invite students to change the microworld’s underlying model so as to modify the objects’ behaviours as they express their personal ideas.

In order put into effect their ideas and accomplish their goals, the students will use the environment’s symbolic representation system. The deep structure access the students have in the microworld’s functionalities will enable them to edit and appropriate the equations already assigned to objects and to construct equations that would depict behaviours which are not accurately described by the existing equations.

As students edit existing equations and conceptualize and develop new ones, we expect them to construct meanings about the role of both the original equations they edit and the new equations they construct.

Criteria:

The soundness of the hypotheses will be attested through the data and the analysis performed with regard to the RQ. The RQ concerns the ways in which students construct meanings about the role of the equations as they interact with the MoPiX DDA and their peers. In order to confirm our hypotheses we will seek for episodes in which the students’ construction of meanings about the role of the equation is achieved as they:

- Associate the symbolic representation to the graphical representation
- Edit already existing equations or by construct new ones.

Confirmation of the Hypotheses:

Hypothesis 1:

The students would construct meanings about the role of the equations as they connect the symbolic representation to the graphical representation available in MoPiX.

Performing the analysis with regard to our RQ we identified incidents in which students’ construction of meanings about the role of the equations was sustained by the connections they made between the symbolic representation (i.e the equations) and the graphical representation generated by the model’s execution (i.e the animated model).

The soundness of the hypotheses was confirmed as we detected specific episodes in which students:

- Attributed meaning to an equation -or certain of its symbols- just after adding it to or removing it from an object and observing the animation.
- Verified the role of an already existing equation or the role of newly formed one after they added it to an object and observed the animation,
- Decided on further changes on a newly formed equation regarding its structure or content as they observed the animation generated after adding it to an object.

Hypothesis 2:

The students would construct meanings about the role of the equations by editing already existing equations or by constructing new ones.

The confirmation of this hypothesis was attained by the analysis we conducted for the CRQ. The analysis indicated the existence of several incidents in which students constructed meanings about the role of the equations as they edited an existing equation or constructed a completely new one.

With regard to the RQ, we classified the students' achievements into the following categories of analysis:

- Construction of meanings about the role of an equation through the interpretation of its symbols.
- Construction of meanings about the role of an equation through the editing of its symbols.
- Construction of meanings about the role of an equation through its conceptualization and development.

The second and the third category of the analysis confirm the soundness of our second hypothesis.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

How do students construct mathematical meanings about the role of the equations while using the available representations in MoPiX to construct virtual models in the context of engaging in engineering design activities?

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

One of the main educational goals presented in the MoPiX Teaching Experiment Guidelines concerned the students' construction of meanings about the role of the algebraic equations and the relationships between them in the context of changing a half-baked microworld. Students used MoPiX built-in and created MoPiX compatible equations so as to ascribe properties and behaviours to their objects and represent phenomena, such as collisions and motions.

On the basis of this educational goal and in order to answer our RQ, we classify the students' achievements into the following categories of analysis:

- Construction of meanings about the role of an equation through the interpretation of its symbols.

- Construction of meanings about the role of an equation through the editing of its symbols.
- Construction of meanings about the role of an equation through its conceptualization and development.

4. ***Construction of meanings about the role of an equation through the interpretation of its symbols***

At first place, students used the equations available in MoPiX without attributing any meaning to the symbols on the left or the right part of the equation. The criterion for selecting and using an equation was plainly its name. For example, students used the equation "amIHittingGround(ME,t)= (y(ME,t) ≤ (height(ME,t)÷2)) and Vy(ME,t)≠0" and the "amIHittingASide (ME,t) = (x(ME,t) ≤ 0 or x(ME,t)≥799) and Vx(object_3,t) ≠ 0" (faulty) presuming – solely judging by its name on the left part and ignoring all the other symbols- that it would make their object "hit the ground" and "hit a side" respectively.

S2	*We want to ascribe the one that makes the ball "Hit a side" [They search the Equations Library and they find the "amIHittingASide" equation which they ascribe to the object]*
S1	*Ahhh… Is it in?*
R1	*It says so. [Showing the environment's respond phrase]*
S2	*Ahhh. Great!*
S1	*I am hitting…*
S2	*«ME»? Me? What is that? HittingASide…. Ok. It's fine.*
S1	*Am I hitting ground? I'm hitting the ground, as well.*
S2	*The ground…. Where is this?[They search the Equations Library for the "amIHittingGround" equation]*
S2	*That's it? That's it or… Oh, no. That's it. [They ascribe the "amIHittingGround" equation to their object]*
R2	*Great.*
S2	*Ok. Then? Let's go ahead. Is there anything else? [They move on to the next category of equations]*

Extract 1: Using the equations without attributing meaning to the symbols

Although they detect the existence of other symbols in the equation they use (i.e the "ME" symbol), students don't seem willing to make any attempt to attribute meaning to those symbols. They continue with the construction of their model without paying any attention to the rest of the equation symbols.

The next step in the construction of meanings about the role of the equations emerged when students started using equations after having attributed meaning only to certain of its symbols. In the case of the "Vx(ME,t)=Vx(ME,t-1)+Ax(ME,t)" equation, students didn't take into account the symbols on the right part of the equation. The decision to attribute it to their object was the result of a comparison between the left part of the equation at hand and the left part of the "Vx(ME,0)=3" equation. After attributing meaning to the symbol of "0" in the latter equation and using it to describe the object's initial velocity, the students sought for an equation to describe the object's velocity at any time. Since the left part of the "Vx(ME,t)=Vx(ME,t-1)+Ax(ME,t)" seemed to meet their needs, students decided to ascribe it to their object regardless of the meaning conveyed in the symbols on the right part of the equation and its structure (i.e the "Vx(ME,t-1)+Ax(ME,t)").

R2 Great. In the first equation instead of "t", what do we have? [In the Vx(ME,0)=3 equation]
S1 The "0".
R2 That does this "0" mean?
S2 That time is 0? No…..
S1 That you don't define the time in this case.
R2 Ok. If I told you to talk about some other time here…. Some other second
S1 Yes?
R2 What would you do?
S1 We would say "with some velocity" meaning… [they attribute the Vx(ME,t) = Vx(ME,t-1) + Ax(ME,t) equation to their object]

Extract 2: The attribution of meaning only to certain equations symbols

Finally, students started using equations after having analyzed the meaning of each one of their symbols and defined each symbol's specific role in the equation. In this case students viewed the equations as sets of symbols that combined into a unified whole, before determining the kind of behaviour it would attribute to their objects.

5. ***Construction of meanings about the role of an equation through the editing of its symbols.***

The second category of achievements refers to the construction of meanings about the role of an equation through the editing of its content. By "editing the content of an equation", we mean the process in which students performed changes to the symbols composing an already existing equation but left the structure of the original equation intact.

Students edited the already existing equations for two distinct reasons: so as to attribute meaning to certain symbols of the equation after comparing the effect that the new equation had on objects with the effect of the original one and -after having attributed meaning to all of the equation symbols- so as to express their ideas and generate behaviours for their objects that were not accurately described by any of the already existing equations. The elements that the students often altered in an equation were the arithmetic values present on its left or right part. The arithmetic value editing they performed could be classified into two categories: editing so as to replace the existing arithmetic value with a different one and editing so as to replace the arithmetic value with a variable.

The students of the 3rd workgroup, after using the MoPiX Library equations to define their object's motion in the horizontal axis, they sought for equations that would make their objects move in the vertical axis. The first equation they detected at the Library was the "Vy(ME,0)=0", an equation that describes the initial vertical velocity of the object. After attributing the equation to the object and watching the animation generated, students decided that the equation they had chosen wouldn't move their object for two reasons. The first one concerned the arithmetic value on the right part of the equation. The "0" had to change into "3", so as for the object to have a velocity in the Y axis.

S2 Press "Play". You didn't do anything. You just made the velocity 0 at the 0 time instance. Its initial velocity is 0. You did nothing to it. It *didn't change, to move downwards [The motion of the ball is exactly the same as the one before attributing the "Vy(ME,0)=0" equation to their object.]*

S1 Yes, yes.

S2 That's what I'm saying. Change it. Give it some initial, we should give it an initial velocity. Isn't it better?

R2 Whatever you like.

S2 Give "3" as an initial velocity. The equation you used before, with the difference that after the equal sign, we will place a "3". There, move it up. [He takes the "Vy(ME,0)=" equation and places it in the Equations Editor. He turns it into "Vy(ME,0) = 3"]

Extract 3: Changing an arithmetic value

The second one became apparent after attributing the "Vy(ME,0)=3" equation to the object and concerned the arithmetic value on the left part of the equation. The "0" value on the left part that referred to the time instance had to change and so as for the object's velocity to be "3" at the following time instances as well. As students looked for ways to incorporate the "all the next time instances to come" element in their equation, they decided that they needed a symbol which they would "just look at and understand that it represents the infinity". The equation they formed was the "Vy(ME,t)=3".

S2 That means that we have to express the "illimitably".

S1 Time… something. Always plus 1.

S2 Do we need a symbol for this?

R2 Do we need a symbol? It's a good question. How do you plan to express it?

S2 With symbols. We usually express something that we can't describe accurately with symbols.

S1 Plus… t. [He writes down Vy(ME,t) = 3].

S1 So when I see this symbol [meaning the"t"].

S2 and I know it represents the infinity

Extract 4: Introduction of a variable

6. ***Construction of meanings about the role of an equation through its conceptualization and development.***

The third category of achievements refers to the construction of meanings about the role of an equation through its conceptualization and development. The difference between this category and the previous one lies in the fact that, in this case, students didn't just change an already existing equation but actually constructed an equation from scratch, using the MoPiX mathematical formalism. This means that in order to express their ideas about the behavior they would like to give their objects, students invented new symbols to which they attributed meaning and related these new symbols to already existing ones, forming a completely new equation. As it becomes apparent, in this case, students not only determined the content of an equation (the kind of symbols they would include), but they also defined the equation's structure (the ways in which the symbols would be related to each other).

The students of the 1st workgroup decided that they would like to link two of their microworld's objects and make them interact under certain circumstances. The idea was to create two equations that would oblige one of the objects to respond to specific events handled by the user. The students decided both on the event that would force the object to respond (i.e the change in another object's position) and on the kind of the reaction such an event would cause (i.e changes in the object's colour). In this process students not only determined the content of the equation (the kind of symbols they would include) but also defined the equation's structure (the ways in which the symbols would be related to each other). Moreover, since no other symbol could describe the effect they would like to generate, students had to invent new symbols to which they attributed meaning, defined the values they would accept and used them so as to relate the new equations to each other.

At first place, students decided on the kind of the behavior they wanted to give to their object. The main idea was that "the ball should change its colour according to the ellipse's position on the Stage". Since students had developed a familiarity with the MoPiX environment they already knew there was no such equation in the Equations Library. They had to build a new one so as to express their idea in the MoPiX formalism.

The first equation developed for this reason was the one that described the condition under which the ball would respond and thus change its colour. Talking about how they would achieve this goal, they decided to include in their equation the Y coordinate of each object and link those Y coordinates so as for the ball to know "I am below now" [meaning below the ellipse].

S1	*Excuse me… The x, y coordinates. Can't the environment recognize them? Their values. Where the objects are situated. Can't it recognize them?*
R1	*Yes.*
S1	*It can recognize them. So I can say that I want this [the ball] to change colour*
R1	*Yes?*
S1	*When it is situated in a Y below the Y of this one for example [the ellipse]*
R1	*You know… I'm thinking… Will the ball know when it is below or above the ellipse?*
S2	*That's what we will define. We will define the Ys.*
S1	*This. The: "I am below now". How will we write this?*
S2	*Using the Y. Using the Y. The Y. That is: when its Y is 401, it is red. When the Y is something less than 400, it's green! Got that? And the velocity. When the velocity has these values, this and this thing happen.*
S1	*Let's start on that. Let's do it.*

Extract 5: Conceptualizing a new equation

Having conceptualized the effect they would like the new equation to have on the Stage's objects, students decided about two distinct elements of their equation. Its content (i.e the symbols it will include) and its structure (i.e the ways in which those symbols will be related to each other). The Y coordinate of each object will be a part of the equation and the relationship between them with be defined by a "less than" sign.

The students proceed by constructing two equations that are related to each other. The "gineprasino(ME,t) = y(ME,t)≤274" and the "greenColour(ME,t)=(not(gineprasino

(ME,t)) × 0 + gineprasino(ME,t) × 100" equations are defined by the students in terms their symbols and their structure.

The students' achievements described in the aforementioned categories could not be viewed independently of the use of the representations available in MoPiX. In order to construct meanings about the role of an equation, students used the DDA's symbolic representation system (i.e MoPiX equations) in the process of:

- Interpreting the role of certain symbols in an equation or interpreting the equation itself as a unified whole,
- Editing the symbols of an already existing equation (modifying the arithmetic values present in the equation and replacing them with another arithmetic value or a variable),
- Constructing a new equation (conceptualizing and developing an equation from scratch, deciding on its structure and content).

In each one of the processes presented above, students, apart from using the symbolic representation system, also used the graphical one. The graphical representation generated by the execution of the equations attributed to the objects was not used so as to directly express ideas as it was the fact for the symbolic representation system, but it was used so as to:

- Attribute meaning to an equation -or certain of its symbols- after adding it or removing it from an object,
- Verify the role of an already existing equation or the role of newly formed one,
- Decide on further changes on a newly formed equation regarding its structure or content.

In any case, the two MoPiX representation systems were used interchangeably by the students in the process of changing a half-baked microworld and both contributed to the student's construction of meanings about the role of the equations.

The description of the students' achievements classified in categories of analysis on the basis of the "educational goal" and the description of the ways in which the representations were used by the students delineate the "modalities of use" (i.e how students used the DDA). However, students' achievements are also related to the "DDA's characteristics". The deep structural access that the MoPiX environment allows the student to gain in order to change the microworld's functionalities (i.e edit/construct equations) and the linked representations (i.e the graphical representation is generated by the execution of the symbolic one) constitute two of the DDA characteristics that seem to influence "the modalities of use" as mentioned above.

3. SPECIFY: - THE KIND OF DATA YOU ANALYSED; - THE SPECIFIC ELEMENTS OF OBSERVATION.

The data we collected during the experimental process and analysed were:

- Audio and video recordings (deriving from a screen capture software for the inter workgroup communication and from a camera/voice recorder for the intra workgroup communication),

- Students' notes and answers on the Work Sheets we provided at certain phases,
- Students' MoPiX models (DDA files saved on MathDiLS),
- Researchers' field notes.

The specific elements that we observed were:

- The students' interaction with the computational environment,
- The students' interactions with their peers (members of the same or other workgroups), students' interactions with the teacher/researchers.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ.

   IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

The wording in the reformulation of the CRQ indicates that we expected students to construct mathematical meanings about the role of the equations as they used the representations available in MoPiX. In order to answer this RQ it was essential to monitor and analyse the ways in which the students interacted with the computational environment (i.e the representations and functionalities available). This specific element of observation gave us a clear view on the ways students used the MoPiX representations and made explicit the connection between the use of representations and the construction of mathematical meanings.

During the experimentation process students interacted both with the computational and the social environment. The social orchestration of the experimentation process gave students the opportunity to interact with their peers while working together as members of the same workgroup and while discussing, sharing ideas and artefacts with members of other workgroups. Since students engaged in joint decision-making processes, shared ideas, developed strategies on which they negotiated and argued, the element of the interaction with the social environment was crucial in order to understand they ways in which the mathematical meanings emerged. The interaction with the researchers/teachers was also an important element since it gave us and understanding about how the researchers'/teachers' input (if any) had an effect at the students' construction of mathematical meanings.

Although we didn't use a specific element of our theoretical framework in the process of the analysis, the theoretical frames and constructs that we adopted for the experimentations with MoPiX continued being a point of reference for us. Thus, drawing on the constructionist and the socio-constructivist framework, we decided to search for ways in which students would "construct" meanings while interacting with the computational media and the social environment. However, the students' construction of meanings can not be viewed outside the context in which it occurs. Since we decided to use for the implementation of our PP a half-baked microworld which by its own nature is designed for instrumentalization and provides students deep structural access so as to be able to change its functionalities, it was apparent to us that the actual process of changing the microworld would constitute an interesting venue for the students' construction of meanings.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

With regard to the characteristics of the DDA our analysis was guided by:

a.3 concerns about the ways representations can be acted on

a.4 concerns about the evolutive characteristics of representations

a.5.1 concerns about interactions between different representation systems within the DDA

With regard to the educational goals our analysis was guided by:

b.1 epistemological concerns

b.2 semiotic concerns

b.5 social concerns

With regard to the modalities of use our analysis was guided by:

c.2 concerns about the functions to be given to the DDA and their possible changes

c.3 concerns about semiotic issues

c.5 concerns about social organization and interactions

For the analysis process the concerns that we took into consideration were no different than the concerns that we regarded as important for the design of the teaching experiment and the concerns on which the reformulation of the RQ was based. As it was the fact for the elements of our theoretical framework, we didn’t explicitly use specific concerns in each one of the analysis phases but rather kept them in mind throughout the analysis process and referred to certain of them when we considered it necessary.

For example, in the first category of analysis that concerns the construction of meanings about the role of the equations through the interpretation of its symbols, we made no reference to the “functions to be given to the DDA and their possible changes” although we *were* interested in the modalities of use. The way in which the symbolic representation system was used by the students (concerns about semiotic issues) was a concern that seemed to be more appropriate for this phase of the analysis process. On the other hand, in the last category of analysis that concerns the construction of meaning about the role of the equations through their conceptualization and development, apart from the semiotic issues (i.e how students invented symbols and attributed meanings to them), we were particularly interested in the ways students acted on the representations, deployed the available representations systems’ interactions (concerns referring to the characteristics of the DDA) and eventually changed the half-baked microworld’s functionalities (concerns about the functions to be given to the DDA and their possible changes).

A.5.6 Analysis of IoE TE with MoPiX

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The envisaged educational goal of the teaching experiment was the development of students' concepts of motion in accordance with Newtonian laws. In the implemented pedagogical plan, this focused primarily on the development of concepts of velocity and acceleration. Specifically:

- velocity as change in displacement
- velocity (in a plane) as a two dimensional vector, either (magnitude, direction) or (horizontal magnitude, vertical magnitude) - the second of these being most naturally encoded in MoPiX notation
- velocity remains constant unless acted upon
- acceleration as change in velocity
- acceleration as a force - specifically acceleration applied at an instant

Through the course of the experiment, students' ways of talking and writing about velocity and acceleration changed in ways consistent with this educational goal, though their use of acceleration was much less secure. Their use of MoPiX showed that they were able to operate with these concepts in order to build models that moved in ways compatible with their intentions, though the nature of this varied between students and achievement was uneven. We would not claim that all students achieved to the same extent. The types of achievements we consider relevant include:

a) Separate treatment of horizontal and vertical components of velocity and acceleration in order to describe motion.

By later sessions, students' problem solving processes while using MoPiX consistently dealt separately with vertical and horizontal components of motion when adding and editing equations to models. Moreover, when using other modes of communication, students also described motion in terms of x and y components, making use of the terms Vx (or 'x velocity') and Vy and, to a lesser extent, Ax and Ay. As may be seen in example 1 below, this allowed descriptions of motion that were more analytical and consistent with the principles identified above.

Example 1 The following task was given both in the written pre-questionnaire and in the post-questionnaire:

> Imagine throwing a tennis ball against a wall. Describe n words how the ball moves and how its motion changes.

Art responded to the pre-questionnaire task:

> The ball flies towards the wall losing height then it hits the wall losing some energy to the wall out as sound, bounces off the wall continues falling but in a different direction.

and to the post-questionnaire task:

> As it is flying towards the wall its x velocity doesn't change while the y velocity is decreasing. When the ball hits the wall the x velocity changes direction (becomes negative)

and some energy is lost to the wall, the y velocity keeps decreasing at the rate of -9.8. As the ball hits the ground y velocity changes direction

Art's responses before and after the teaching experiment show some similarity in the use of the idea of 'flying' towards the wall and losing energy to the wall (a concept presumably drawn from his lessons in Physics as his use of MoPiX had not included this phenomenon). However, his response to the post-questionnaire (i) presents velocity as a vector quantity, separated into horizontal and vertical components (ii) recognises that the horizontal velocity does not change until it hits the wall (iii) identifies bouncing off a vertical or horizontal surface as a change of sign of the horizontal or vertical velocity respectively (iv) recognises that the vertical velocity is affected by the constant acceleration of gravity.

b) Development and use of the concept of acceleration is more fragile than that of velocity.

Students quickly developed systematic strategies to construct models involving only velocity, analysing the values needed to produce the desired effects. In general, they struggled to solve problems involving acceleration and were inconsistent in the ways in which they talked about it and applied it. This may have been at least in part because acceleration was addressed later in the teaching sequence. Example 2 illustrates the difference.

Example 2. During Session 5 students were able to use changes in velocity in order to change the direction of motion of objects. In Session 7, they were asked to achieve changes in direction by applying an acceleration at an instant. Aa chose first to work on the problem of drawing a square using changes in velocity in order to change direction, he then revisited the same task of drawing a square by using acceleration as a force applied at an instant in order to achieve the same effect. In each case, Aa started by using a trial and improvement approach in order to make the first corner but then used systematic methods to turn subsequent corners. When using velocity, his progress through the trial and improvement stage was rapid, using systematic methods to correct errors. The only errors made on turning subsequent corners were errors of sign and by the final corners he was changing both horizontal and vertical components of the velocity without making intermediate trials. When using acceleration, the initial trial and improvement stage was much longer, involving a high number of trials, some of which did not appear systematic. Having achieved the first turn, his methods appeared more systematic but much slower than when using velocity directly. Towards the end of this task, he spent several minutes carefully examining the set of equations, pointing repeatedly to the velocity equations as if recalculating the horizontal and vertical velocities at each application of an acceleration. We interpret this as an indication that Aa was just beginning to operationalise the concept of acceleration as change in velocity.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

We hypothesised that, through use of MoPiX within the proposed pedagogical plan:

- the process of defining the behaviours of objects and observing the consequent behaviours would enable students to form and test their own hypotheses about the relationships between formal definitions and observed motion and hence to construct conceptions of motion consistent with Newtonian laws;

- the use of multiple semiotic systems, each affording different meaning potential, would provide students with greater opportunities to communicate effectively about motion and hence to construct and to interpret representations of motion in accordance with Newtonian laws;
- the social and technological environment and the encouragement of various forms of communication would provide opportunities for exploratory talk, encouraging students to engage in higher level reasoning and argumentation and to use explicit forms to represent their ideas.

We proposed the following criteria:

- *Students successfully make use of equations to construct animated models whose behaviours fulfil requirements posed by tasks set in the pedagogical plan or posed by the students themselves.*

All students succeeded in making use of equations to construct animated models. Most tasks set by the pedagogical plan were either completed successfully or were adapted by students in accordance with their own objectives. Students became adept at constructing objects with straight-line motion with a desired direction and velocity and were able to use changes in velocity to change direction. Behaviours that involved use of acceleration were more difficult to achieve. Behaviours involving interaction between objects proved to be frustrating to achieve because of the difficulties in using the editor in version 1 of MoPiX.

- *Students interpret graphs of motion, relating these accurately to the behaviours of their animations.*

Changes made to the pedagogical plan in response to contextual factors prevented us from fully implementing the part of the plan related to graphing. Only one student attempted the ‘four graphs’ task. He was successful in interpreting each of the graphs. However, we cannot take this as adequate evidence of effects of the MoPiX environment.

- *When communicating verbally among themselves or with a teacher or researcher, students describe, predict and analyse motion in ways consistent with Newtonian laws.*

Interestingly, the consistency of verbal communication about motion varied according to the register used for the communication. When using terminology derived from the MoPiX notation (e.g. ‘x velocity’ or ‘vertical acceleration’), description, prediction and analysis of motion was generally consistent with Newtonian laws. When using more ‘everyday’ terminology such as ‘speed’, ‘getting faster’, there was often slippage into inconsistent, intuitive or everyday ways of talking about motion. For example, Z, describing the motion of a ball thrown into the air:

Z as it goes up it gets slower until at the top it’s zero

CM then what happens to it?

Z it gets faster

This way of speaking did not help her to understand the action of acceleration on the ball. Shortly afterwards, however, when asked to consider what happens to Vy , she appeared to experience an ‘aha!’ moment:

Z it stops

CM and then what happens to Vy

Z it gets negative … Oh yes!

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

What concepts of motion are represented through students' semiotic activity in the context of use of MoPiX?

How do students operate in MoPiX with the variables x and y, Vx and Vy, Ax and Ay? In order to achieve what goals?

What forms of language and other modes do students use to communicate about velocity and acceleration as they work to construct animations and to interpret sets of equations and graphs?

What interpretations do students make of connections between animations and graphs of aspects of the motion of the animations? (We are unable to address this part of the RQ as unanticipated contextual factors meant that we were not able to complete the part of the pedagogic plan related to graphs.)

What choices do students make between and within the semiotic systems offered by MoPiX and the context of its use in order to communicate meanings related to motion?

Do students' communications about velocity and acceleration outside of MoPiX vary through the course of their experience with MoPiX?

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

Vector notions of velocity and acceleration

Vector notions of velocity and acceleration (with horizontal and vertical components defined separately) became well established in students' activity within MoPiX and in other modes. Students' problem solving processes while using MoPiX consistently dealt separately with vertical and horizontal components of motion when adding and editing equations to models. This separation is supported by the structure of the semiotic system of MoPiX equations. Moreover, when using other modes of communication, students also described motion in terms of x and y components, making use of the terms Vx (or 'x velocity') and Vy and, to a lesser extent, Ax and Ay.

Relationships between the values of horizontal and vertical components of velocity were coordinated in order to determine the direction of motion. This was done most efficiently when motion was horizontal or vertical and when changes in direction were perpendicular, using the strategy of swapping x and y component values and changing signs. Other cases were also solved successfully (e.g.. drawing a triangle) but in these cases qualitative evaluation of the graphical output played a more important role, apparently needed in order to validate quantitative analysis of components of velocity.

Changes in the values assigned to velocity and acceleration were used in order to effect changes in the motion of animated objects, focusing both on speed and on changes of direction. After early tasks introducing students to the basic sets of equations required to make an object move, a common strategy used in constructing new objects was to apply this basic set rapidly and then to select and edit the equations, setting new initial values for velocity and acceleration or effecting changes in velocity or acceleration at specific times. The equations effecting change in displacement x(ME,t)=x(ME,t-1)+Vx(ME,t) and change in velocity Vx(ME,t)=Vx(ME,t-1)+Ax(ME,t) were generally not edited but were used more as ‘black box’ equations although, when attention was drawn to them explicitly in interaction with the teacher/researcher, students were able to explain their functioning.

The oral language used by students through the course of the teaching experiment increasingly made use of component related terms derived from the MoPiX language (e.g. ‘x velocity’). This supported effective problem solving and description of motion consistent with Newtonian principles. It was, however, more consistent and secure when dealing with velocity than with acceleration. The use of ‘everyday’ language and gesture provided resources for communicating about acceleration that were not consistent with MoPiX formalism or did not support effective problem solving.

As students made use of various modes of representation (‘everyday’ language, specialist language of mathematics or of MoPiX, MoPiX programming, pencil and paper writing, drawing, MoPiX animation, gesture) different sets of semiotic resources and consequent meaning potentials were available to them. These influenced the focus and direction of their activity. For example, a task that was originally posed as the creation of a pattern of lines, through making use of the resource of colour provided by preparing the design of the pattern using the Paint software and interaction using gesture with a drawing, became a task to draw a ‘firework’. This changed the way in which MoPiX lines functioned in the students’ talk and within the task. From being a matter of creating lines (static objects resulting from a motion) the goal of their activity became to create an animation in which the lines were traces of motion rather than objects in their own right. In other cases, moving their attention to symbolic resources in pencil and paper mode or in conjunction with calculator use changed the nature of the problem students were attending to from qualitative evaluation or description of the shape of a graph or trace of an animation to quantitative evaluation or calculation of the various aspects of motion.

c) Operationalisation of the concept of acceleration as change in velocity appears to be supported by some forms of semiotic resources more than by others.

Students’ ways of talking about velocity and acceleration and their use of these in problem solving varied across the course of the teaching experiment and across the various modes of communication in use. This aspect is still subject to fuller analysis but we present example 3 here to illustrate the way in which different modes of communication may affect the meanings constructed for acceleration.

Example 3. While working on question 3 of the post-questionnaire (see below), Ab and Aa made use of the diagram provided, interacting with it with speech and gestures. They also made use of a calculator, pencil and paper and MoPiX. When using the diagram, they struggled with the idea of constant acceleration, which seemed to conflict with their interpretation of the diagram. Ab seems to confuse acceleration with velocity:

> it's decelerating here [slides from t=50 to t=130 LH] then here it's zero here [points LH and RH at t=130 (prolonged)] and starts accelerating again [rapid slide from t=130 to t=150 RH]

The diagram and interaction with the diagram using gesture to mimic the imagined motion of the ball provided resources that did not enable the students to distinguish clearly between acceleration and velocity. They did not distinguish between horizontal and vertical components and associated upward movement with acceleration, even moving the sliding finger faster as it moved upwards.

As they started to fill in the table, however, renewed interaction with the wording of the question led them to fill the Ax column with zeros and the Ay column with -0.1 all the way down. Re-visiting the wording of the question prompted the students to separate acceleration in the horizontal and the vertical directions and to operate with them as constants. The use of the verbal and symbolic modes rather than the diagrammatic enabled them to complete the acceleration values in the table correctly, apparently in contradiction to their earlier ideas.

After an initial attempt to complete the Vy column by considering the diagram, they decided to calculate instead. Aa got out his calculator and prepared to do some calculations. With the calculator by his side, he developed the approach he intended to take, communicating with his partner in interaction with both table and diagram:

> if you got y acceleration at -0.1 here [points to Ay at t=0 in the table] to find out at what point it stops here [points to t=50 on diagram] if you times that [points to Ay at t=0 in the table (-0.1)] by the time taken to reach here [points to t=50 on diagram] .. you should get the velocity for the y

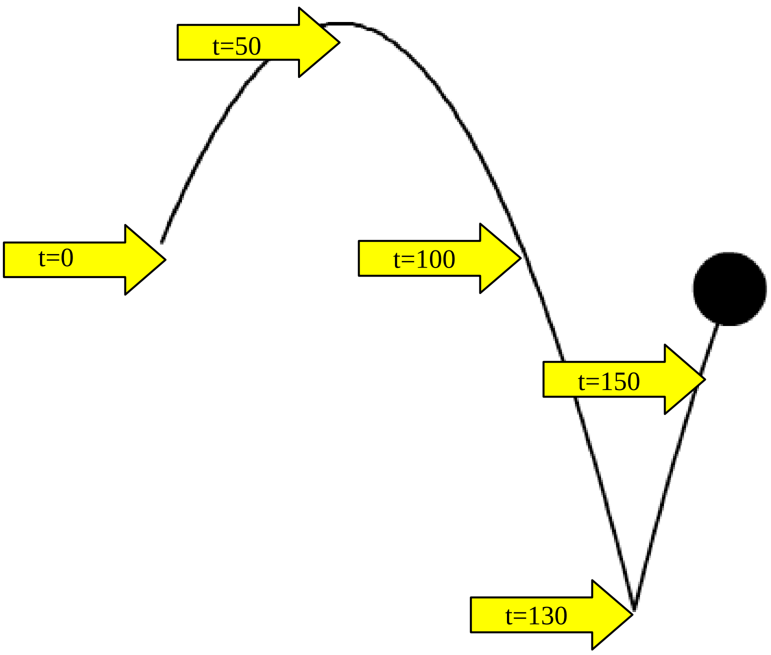
Having decided to calculate, the affordances of the calculator itself allowed connections to be made between, on the one hand, the symbolic mode of the table and, on the other hand, the diagram. Pointing at the diagram now served to identify a point in time, rather than a movement.

This episode illustrates the fragility of the notion of acceleration for these students. It was only with the support of a range of interacting semiotic resources that they could be successful in resolving the problem.

3. The diagram below shows the path of a ball thrown into the air and then bouncing off the ground.

   The ball's initial velocity (at time t=0) is 2 in the x-direction and 5 in the y-direction. Its acceleration is -0.1 in the y-direction (a MoPiX approximation for gravity).

   Complete the table below with the velocity and acceleration of the ball at the given times.



time	velocity		acceleration	
	Vx	Vy	Ax	Ay
t=0	2	5	0	-0.1
t=50				
t=100				
t=130				
t=150				

In the language of didactic functionalities, the characteristics of the DDA provide a representation of motion separated into horizontal and vertical components. This addressed the educational goals directly by providing students with an alternative language that moved them away from ‘everyday’ ways of representing motion, enabling analytic and quantitative approaches to defining and describing it. The provision of a library of equations allowed students to develop a strategy of paying attention to the meaning of a limited set of equations while using others in a ‘black box’ mode, focusing both attention and effort on specifying the aspects that would effect desired changes in speed or direction. Developments to MoPiX 2.0 are intended to further facilitate this ‘layered’ approach.

We have identified a duality in the interpretation of graphs as static patterns or as traces of motion. The graphical representations of MoPiX support both these interpretations while the symbolic representations support the second, dynamic interpretation. This dynamic interpretation seems more aligned with our educational goals but students’ adoption of it is affected by the modalities of use, in particular by the other forms of representation provided in written materials and in interaction with the teacher and by the alternative semiotic resources available in the immediate context.

3. SPECIFY: - THE KIND OF DATA YOU ANALYSED; - THE SPECIFIC ELEMENTS OF OBSERVATION.

Up to this point, the main data analysed have been extracts of video data involving pairs of students working together on MoPiX related tasks. The video focused on the area in front of the students, encompassing the screen, hand gestures and pencil and paper. This has been supplemented by pencil and paper work and MoPiX models produced by students during the course of the extracts analysed. Selected extracts have been transcribed, distinguishing in parallel the different modes of communication in use. Thus each transcription is structured to indicate simultaneously at any point in the extract: speech of the participants; interaction with MoPiX (e.g. dragging, editing, authoring equations, playing or stopping an animation); screen display; gesture; pencil and paper production (including use of computer based analogues of pencil and paper, e.g. Paint).

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ. IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

Extracts have been chosen for analysis on the basis that they appeared to be of interest in relation to addressing some aspect of the Re-CRQ. In particular, because of our interest in semiotic activity in different modes, we have chosen episodes in which students have made use of several modes of communication. We also selected episodes in which there was interaction about velocity and/or acceleration between students and/or with the computer. The process of selection of extracts for detailed analysis is iterative. Having identified preliminary results we intend to revisit the data to identify further evidence related to our answers.

Our social semiotic theoretical framework orients us to consider the meaning potential of the semiotic systems in use within the context of situation and to interpret the texts produced by students (in any mode), taking into account both the context of situation and the broader context of culture. At an early stage in the analysis of chosen extracts, transcripts were divided into ‘moments’ of communication that were considered to have some meaningful coherence; this division was a pragmatic consideration without an explicit theoretical basis, though our identification of ‘meaningful coherence’ is based on an assumption that our own understanding of the context draws on similar sets of discursive resources to those of the students.

The process of analysis involved both the application of a priori categories and the iterative definition and refinement of categories derived from the data, accompanied by ongoing interpretation of patterns in the coding. Initial strands of coding include:

1. Mode (this could be a multiple coding where several modes were relevant to the interaction during a single moment)

2. Form of reference to velocity and/or acceleration (this strand reflects both semiotic and epistemological concerns)

3. Static or dynamic interpretation of MoPiX graphics (this strand is an example of one that emerged from interaction with the data)

4. Strategy for selection of equations

Having coded moments of the transcript, patterns and relationships are sought within and between strands. In particular, given our concern with choices between modes and the relationships between semiotic resources and meanings, we examine relationships between strand 1 and others.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

With respect to the characteristics of the DDA, our analysis is guided by the following top concerns:

- concerns about the ways mathematical objects and their interaction are represented
- concerns about the ways representations can be acted on

With respect to educational goals, our analysis is guided by the following top concerns:

- epistemological concerns
- semiotic concerns

With respect to modalities of use, our analysis is guided by the following top concerns:

- concerns about the functions to be given to the DDA and their possible changes
- concerns about semiotic issues
- concerns about social organization and interactions

Our analysis does not attend separately to each of these concerns but, by examining patterns and connections across strands, addresses relationships between them. Our focus on semiotic resources brings us to identify in the analysis not only how the representations provided by the DDA are used within MoPiX but also how they are taken up in verbal language and how they relate to other representations available in the context of situation.

A.5.7 Analysis of IoE TE with MaLT

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

Our Educational Goal was expressed as follows:

> *To develop students' ability to recognise and analyse the properties of three dimensional geometrical shapes through construction and manipulation of 2D and 3D representations.*

Through the course of the teaching experiment, students were introduced to a range of ways of constructing representations of three dimensional objects and were successful in using these in structured and directed tasks. Some also used these spontaneously to support their problem solving or to communicate to others.

For example, in Session 2, students were introduced to the construction of isometric drawings of three dimensional objects.

K's choice of representation was initially naïve, juxtaposing 2D representations of separate faces. Her initial drawing was two dimensional, showing only the face of the shape closest to her. When questioned by the teacher she indicated that she could also see 'a little bit' of another face. She added a two-dimensional drawing of this second face next to the first face, without representing the angle between the faces.

After instruction in isometric drawing, using isometric grid paper, she was successful in drawing simple cuboids but still struggled to represent more complex shapes, again appearing to represent components of a complex shape separately, juxtaposing them joined by an edge rather than a face, without representing their relationship accurately.

In Session 3, students were given the task of constructing two walls of a room using MaLT. Students were instructed before starting to construct in MaLT first to imagine and then to make a sketch of what the two walls would look like. K immediately sketched an isometric representation of a cube (on plain paper), then used her rubber to erase the lines forming the back corner of the cube, leaving a representation of two walls.

This could be interpreted as a progression in K's ability to recognise and represent relationships between the faces of 3D shapes. On the other hand, we must be aware that the change in the context of the task may have affected the resources available to her.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

Our hypothesis was that use of the variety of symbolic and visual semiotic systems provided by MaLT and by the other multi-modal experiences with 3d objects and 2d representations included in the pedagogical plan, students' ability to analyse 3D shapes and their properties would be enhanced. We proposed the following criteria:

- *Students successfully use MaLT to draw 3D objects that fulfil the requirements posed by tasks set in the pedagogical plan or posed by the students themselves.*

All students experienced some degree of success in the tasks set but this was limited by the time available and their difficulties in learning to use the software.

- *Students use the variation tool in MaLT and can explain its effect.*

Because of the context in which we were conducting the study (see below) we were able to make less use of the 2d variation tool than had been anticipated.

In Session 4, students used the variation tool in order to explore pre-constructed models (e.g. joining up prisms by dragging sliders representing angles and/or side lengths) and some were able to connect the values displayed on the variation tool with the movement of the shapes, in particular connecting the number 60 to the angle of an equilateral triangular face of a prism.

Other students, however, although they could successfully manipulate the sliders, had difficulty connecting the numerical display to the shape. In at least one case, this appeared related to insecure knowledge about the angles in a triangle as may be seen in this conversation between teacher (GD) and student. T had called the teacher for help when faced with the task of explaining why the value of 60 on the slider made the triangular prism join up.

1	GD	When you look at just the triangle bit, what type of triangle is it?
2	T	triangular prism?
3	GD	Just the triangle, forget about the fact that it's three dimensions. What type of triangle is it?
4	T	… (*hesitates - no answer*)
5	GD	What can we say about the lengths of the sides?
6	T	They're all equilateral
7	GD	Exactly. It's an equilateral triangle. So what are the angles in an equilateral triangle?
8	T	All the same
9	GD	They're all the same and they have to add up to?
10	T	One hundred and eighty
11	GD	So what's the size of an angle?
12	T	One hundred and eighty
13	GD	They **add** up to one hundred and eighty
14	T	forty five degrees
15	GD	What's three lots of forty-five?
16	T	pardon?
17	GD	What's three lots of forty-five?
18	T	um …
19	GD	Does it make one hundred and eighty?
20	T	… (*shakes head - no*)
21	GD	So what number do we need for three lots of that same angle to make one hundred and eighty degrees?
22	T	… seventy? … sixty? (*very quiet and hesitant*)
23	GD	… So what's the angle in an equilateral triangle? You think about that.
24	T	Is it forty-five? (*more loudly and confidently*)

The apparent failure of this dialogue to help T make a connection between the value 60 and the size of the angle seems related to the discontinuities in the theme of the discussion at lines 14/15 and 22/23. GD changes the theme from angle to calculation and then back again. T’s lack of connection between the two themes is evidenced by her request for clarification “pardon?” at line 16 and by the contrast between her hesitance at line 22 and her confident repetition of her answer forty-five at line 24. T’s difficulty in dealing with thematic discontinuity also seems evident in her lack of any answer after the shift at lines 2/3 between considering the 3D representation in the MaLT turtle screen and considering an abstract equilateral triangle.

In session 7, students constructed procedures with variables to represent the turn of a revolving door or the sliding distance of a sliding door. They then called up the variation tool and used it to animate their doors. There is no evidence that students attended to the numerical values displayed on the variation tool. Rather it appeared to be used simply as a ‘handle’ to manipulate the display. This mode of use is consistent with the overall design purpose of the activity. The task did not demand attention to the values.

- *In their presentations at the end of the project, students make use of different forms of representation for their virtual building that are consistent with one another.*

Each group of students worked to produce a poster displaying their design for the new sports centre and presented this to their colleagues in the final session. They incorporated a variety of forms of representation into their posters, including:

- informal drawings, both of the whole building and of particular features
- 2D plans (using either plain paper or squared paper) of levels of the building and of specific features
- 2D elevations
- isometric drawings (using either isometric paper or plain paper) of the whole building
- MaLT screenshots, showing the turtle screen and the associated procedure, of the moving doors

The narratives given in their oral presentations to the class made some connections between the various components, but in general, the dimensions and shapes of different forms of representation were not consistent. Students thus developed skills in producing specific forms of representation but did not make explicit connections between them. This may be dues at least in part to the lack of emphasis on measurement in the pedagogic plan; each form of representation used its own unit of measure (2cm cubes, 1 cm squares on paper, turtle steps) but no connections were made between these.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

What meanings do students make in relation to three dimensional geometry through their semiotic activity in the context of working with MaLT and other modes?

> *What relationships are there between procedures students write or changes they make to given procedures and the properties of the shapes they are working with?*
>
> *How do students use the variation tool and for what purpose?*
>
> *What interpretations do students make of the effects of using the variation tool?*
>
> *What choices do students make between and within semiotic systems in order to communicate their completed design to their peers? To what extent and in which ways are the properties of shapes represented?*

Before proceeding to answer this reformulated question we must comment on the context in which the implementation of the teaching experiment with MaLT took place. We worked in a state secondary school with tightly constrained curriculum. We negotiated with the school about the use of MaLT and amended our pedagogical plan in order to make strong connections with the curriculum. Some of the work with MaLT had to be arranged out of school time, which added some pressure on students.

The participants were students in Year 8 (12-13 years old) with a low level of achievement, who were not perceived by the school or by themselves as successful learners, with negative consequences for their engagement and classroom behaviour, leading to frequent teacher intervention. Moreover, students had no pre-experience working on Logo, adding more constraints to the implementation. The effect of these contextual elements and of our preliminary engagement with the data collected has been to change the focus of our articulation of the Reformulated Common Research Question in order to take in to account the nature of the students' engagement with MaLT and, in particular, the strong involvement of the teacher, student teacher and researchers in supporting and guiding student activity. The relationship between the semiotic activity of the teachers and researchers and that of the students themselves thus became a significant issue that could not be ignored. We would therefore add a further sub-question to our Re-CRQ:

> *What role did the semiotic activity of teachers and researchers play in shaping students' use and interpretation of the various forms of representation available?*

2. ANSWER YOUR RE-CRQ. WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

What meanings do students make in relation to three dimensional geometry through their semiotic activity in the context of working with MaLT and other modes?

1. The meanings and measurement of angle

The students' prior knowledge and understanding of angle measure in 2D or 3D proved to be fragile. While this did not prevent them from engaging with MaLT it appeared to affect the ways in which they used angle values within Logo commands.

a) angle as arbitrary measure: M and S were attempting to debug the incorrect 'door' procedure provided on a worksheet (session 5). S recognised that all the angles should be the same but did not recognise 90 as right angle

> S I've changed all the forwards to 6
>
> CM Ok
>
> S And now the rights need to be 100 don't they
>
> CM do they?
>
> S or 90
>
> CM which?
>
> S I don't know. I've got two 90s and two 100s
>
> laughs
>
> S I'll go 90 for now and see

This solution worked satisfactorily for S and he shared it with M. The argument given for having all the angles the same is based on the visual feedback and does not analyse the source of the 'wonkiness'.

> S all the rights need to be the same

M Aah

JA why?

S Because otherwise it tilts and it goes all wonky and doesn't work

Later when explaining to M how to correct his procedure:

All your rights need to be the same number, all your forwards need to be the same number

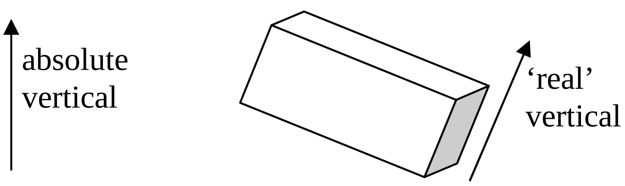
The focus is on the sameness of the values rather than on the values themselves or on the way in which varying the values might change the shape. Indeed, having made all the ‘forwards’ the same, the two boys complained that their ‘door’ looked more like a wall (i.e. was square) but did not diagnose the problem or change the values. The meaning they have for *rectangle* or *square* thus includes the condition that angles should be the same - a condition that is moderated both visually and symbolically (by checking that all numbers are the same), but without a strong association between numerical value and measure.

b) ninety degrees as a building block: Most of the students were confident in using ninety degree angles. These were appropriate and useful for many of the set tasks, which mostly involved rectangular components. During these tasks (with the exception of the aeroplane path task in the introductory MaLT session), whenever a turn was required, 90 was given as the angle of turn and if this did not give the correct outcome, the student would press Insert again to repeat the same turn. This would sometimes be done repeatedly and rapidly, making the turtle spin, until satisfied with the position. This trial and error strategy was used with all kinds of turns, but was especially prevalent when using *roll* commands.

The Logo editor allows any instruction to be re-played by positioning the cursor and pressing Insert again rather than re-entering the instruction. This characteristic facilitates the ‘building block’ approach, allowing some successful constructions. However, because it does not leave a trace of all the instructions that the turtle has followed, it does not lend itself to subsequent review. Students found it hard to remember exactly what instructions the turtle had been given and were hence unable to analyse or to repeat constructions. This mode of use suggests that 90 was being used as a syntactically compulsory element of commands (i.e. *rr (90)* as an indivisible semantic unit) rather than having an independent semantic function.

3. Direction: absolute, ‘real’ or relative

There was a persistent mismatch between the relative geometry of Logo provided by MaLT and students’ use of the terms *up* and *down*. Strong absolute up-down orientation persisted in spite of students’ frequent manipulation of the turtle screen to change their point of view with respect to their constructions. As well as using *up* and *down* with reference to the absolute vertical direction, the terms was also used with reference to the constructed objects. As most of the tasks involved constructing representations of ‘real’ objects: the path of an aeroplane, walls of a room, doors, these constructions could be considered to have their own ‘real’ vertical orientation even when viewed from a position that moved this away from the absolute vertical. Students did not have problems distinguishing between absolute and ‘real’ senses of *up* and *down*. They did, however, frequently use the Logo commands *up* and *dp* when trying to move in absolute or ‘real’ vertical directions.



To what extent are students' constructions in different semiotic systems consistent with one another? In particular, are representations of properties of shapes consistent in different systems?

1. Relationships between different modes during MaLT use

a) 'everyday' language, 'specialist' language and Logo formalism:

We distinguish three different sets of discursive resources used to describe movement: the everyday language of students (*turn*, *that way*, *forward*, *down*), the Logo formalism (*lt*, *fd*, *dp*) and the specialist language used by the teachers/researchers, which serves as a 'translation' of Logo formalism (*pitch*, *roll*, *turn*). Clearly there are overlaps between the three: for example, *forward* and *turn* are used both in the everyday and in the specialist discourse, while there is an intentional close match between the Logo formalism and the specialist language.

However, these terms are not identical in their reference. For example, T appeared to use an 'everyday' *forward* to indicate an absolute direction perpendicular to and away from the screen.

speech	*turtle screen*	*gesture*
Shall I do the wall forward?	first wall complete, turtle facing left of screen on completing bottom side of wall	rt arm horizontal across screen, hand in PDN position, fingers towards body; fingers moved towards body on 'forward'

In this case, her everyday meaning of *forward* did not seem to interfere with her use of Logo formalism as she sought to find the correct type of turn command to orient the turtle in the desired direction. Within Logo, she was clearly able to distinguish that *forward* does not indicate any kind of turn and thus was not a relevant choice for her in this context.

The introduction of the specialist terms *pitch* and *roll* and the associated gestures (and the associated Logo commands *up*/*dp* and *rr*/*lr* provides a new set of resources with no common equivalent in everyday language. *Up pitch* and *down pitch* were adopted by students and often accompanied by the relevant specialist gesture, though also often associating *up* and *down* with absolute reference rather than considering the starting heading of the turtle, as in the following example:

> S: I wanted it to go down pitch so I was intending it to go down this way but instead it turned right and went down that way

Right and *left roll* were less commonly used, especially early in the teaching experiment, except when prompted by teacher/researcher intervention. When they were used, this tended to be in a trial and error way. We hypothesise that this lack of analytic use is related to the lack of a common everyday equivalent that might support students' use of these terms.

b) paper and pencil (isometric representation) and MaLT

Students were encouraged to make use of paper and pencil to support their planning of constructions in MaLT. There were also instances where teachers made use of pencil and paper as part of their interventions to help students. When working on the task of drawing two walls of a room, students generally sketched two walls using an isometric perspective. Some achieved this by sketching a cube and then erasing the extra lines. The subsequent constructions in MaLT were similar (though not isometric) in that they provide a 'framework' of edges. Interestingly, the paper and pencil drawings were not used explicitly for reference while working with MaLT. The two activities appeared to be perceived as separate and self-contained.

The representations in the MaLT turtle screen use 'true' rather than isometric perspective and when drawing rectilinear figures, edges further away are often 'hidden' by edges nearer to the viewer. Students soon discovered how to overcome this problem by dragging the turtle screen in order to change the orientation of their figure. They appeared to prefer this method (which we believe was not intended by the designers) to using the arrow keys to alter their viewpoint. The preferred orientation seemed to be one that coincided as closely as possible with the conventional isometric view. This suggests that the MaLT constructions and paper and pencil isometric drawings may be being used as part of the same overall system of representations.

2. Communicative purpose and perception of activity goals

Our pedagogic plan situated the students' work within the context of the 'real life' project of designing a new sports centre for the school. While skills related to the use of specific forms of representation (plans and elevations, isometric drawings, MaLT procedures and turtle traces) were developed through separate tasks, some of which were not expicitly related to the design project, all sessions were framed by reference to the overall design aim and, with the exception of the introductory MaLT session, included some work related directly to the preparation of the design. Of course, it cannot be assumed that students will make the intended connections between the various parts of the experiences that are offered to them. As noted above in relation to the achievement of educational goals, a lack of emphasis on measurement in the pedagogical plan and the use of non-standard measures in MaLT and other forms of representation may have contributed to a perception of the overall design goal as the presentation of qualitative ideas rather than a quantified blueprint.

What role did the semiotic activity of teachers and researchers play in shaping students' use and interpretation of the various forms of representation available?

1. The use of gesture and the 'playing the turtle' metaphor

During use of MaLT, it was noticeable that the teachers and researchers made extensive use of gestures in an apparent attempt to support students' planning and execution of constructions in MaLT. One significant type of gesture was a set of stereotyped hand and/or arm movements, often associated with use of the terms *turn*, *pitch* (or more frequently *up* or *down*) and *roll* and the associated Logo instructions (see Table 1 for the codes used in transcription of these gestures). For the teachers and researchers, using these gestures as ways of thinking and communicating about the movement of the turtle within MaLT was consistent with the notion of body syntonicity, widely considered a feature of Logo use, and with the metaphor of the user 'playing the turtle'. In the 3D context, it is not possible to physically act out turtle movements with the whole body. Instead, the hand (or, during the initial introduction to MaLT in the teaching experiment, a toy aeroplane held in the hand) substitutes for the body.

Students adopted some of these gestures but used them in ways that were not always consistent with the turtle movements. For example, T, having constructed one rectangular wall, was trying to construct a second perpendicular to it. She explained what she was trying to draw using language and gesture:

1	here	*whole rt arm vertical P0, palm facing away from body, moves up in direction of fingers*

2 turn here	*TR, arm moved in direction of fingers (maintaining TR position)*
3 turn here	*attempt to move rt hand TR again (too difficult?)*
4	*switch to lt hand, arm horizontal pointing rt, hand PDN (fingers pointing down)*
5 turn here	*moves forearm clockwise, hand still PDN (fingers pointing left)*
6 but I want it to come forward	*turns arm (awkwardly) so that, hand still in PDN position, fingers point towards body*

The switch between lines 3 and 4 between use of right and left hands appears to be a response to the physical difficulty of achieving the desired position with the right hand. We consider what remains the same and what is changed as this switch of hand occurs. The switch allows T to maintain the direction in which the fingers are pointing (down). This may be taken to represent the turtle heading within the vertical plane parallel to the screen. However, in switching arms, she changes the relationship between arm and hand from a *turn* gesture to a *pitch* gesture. We use *turn* and *pitch* within the conventions set up by the teachers/researchers and the Logo language, not in order to suggest that T associates her gestures with these terms. On the contrary, she does not appear to attach any significance to the distinction, focusing solely on the position of her hand and the direction in which her fingers are pointing in order to describe the intended turtle movement. While she is to some extent ‘playing turtle’ with her hand, she is defining the turtle’s movements by using position and heading at the corners of her imaginary wall rather than by using turn and distance as required by the Logo language. The use of the turn and pitch gestures is thus not supporting her move into using Logo code and may indeed have made her communication with teachers/researchers less effective.

The use of arm and hand movements to mimic the position and/or movement of the turtle was common among the students. Whereas this might be seen as a way of ‘playing turtle’, it is possible to interpret it as a variety of ‘pointing’ at an imaginary turtle. Certainly at least one student explicitly refused to accept the metaphor offered to her by a researcher:

JA if you imagine yourself as a turtle, how are you going to move?

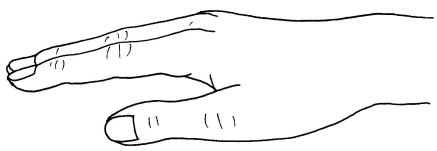
K it is very uncomfortable imagining myself as a turtle ... erm

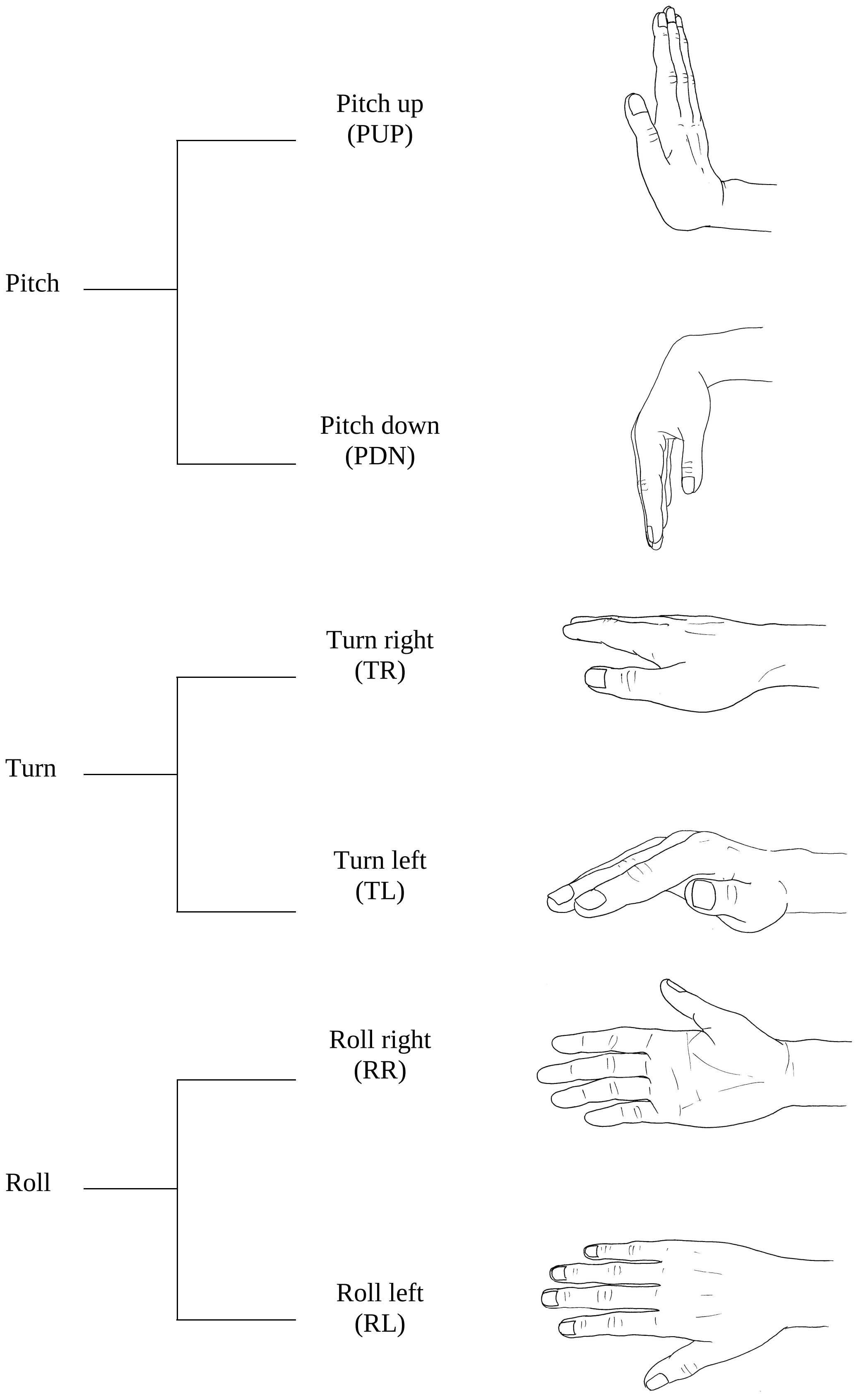
JA or imagine your hand

K I don’t want to be a turtle

Pointing is a widespread form of representation of position, common in everyday discourse. While it might appear at first sight that students adopted the specialised gestures employed by the teachers/researchers, the students’ use and interpretation of these gestures may be closer to the resources of everyday discourse than to the specialised 3D movement representation.

Table 1: MaLT gesture codes

Position 0	_________	P0	


Pitch
Pitch up
(PUP)
Pitch down
(PDN)
Turn
Turn right
(TR)
Turn left
(TL)
Roll
Roll right
(RR)
Roll left
(RL)

3. SPECIFY:

- THE KIND OF DATA YOU ANALYSED;
- THE SPECIFIC ELEMENTS OF OBSERVATION.

The data analysed included:

- paper-and-pencil work collected from students throughout the teaching experiment. These are of three kinds:
    - directed drawing arising during instruction in the use of particular kinds of representation;
    - drawing used to support work on MaLT tasks;
    - drawing and writing related directly to the sports centre design project

  In each of these cases, we observe the form of representation used and the extent to which it matches conventional forms.
- video and audio records of all whole class interaction episodes
  We focus on the modes used by teachers and students to communicate about shape and about the meaning of the various forms of representation, including those provided by MaLT.
- video and audio records of a group of four students working together during ordinary class-based work and of pairs or individual members of this group working with MaLT. These records capture screen content (in the case of MaLT work), gesture and use of paper and pencil.
  We observe:
    - student-student communication about shape (and student-teacher communication in the case of teacher interventions);
    - student-computer interaction when using MaLT, including: Logo instructions; use of the variation tool; dealing with error; gesture.
- saved MaLT procedures
  We focus on the structure of procedures and their integrity as representations of 3d shapes.
- posters produced by all groups in the final session
  We observe the choices made by each group about what to include on their poster, the physical arrangement of items on each poster and the extent of coherence between the representations included.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ.

   IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

In the case of video records of whole class interaction and of group or individual work, extracts have been chosen for detailed analysis on the basis that they appeared to be of interest in relation to addressing some aspect of the Re-CRQ. In particular, because of our interest in semiotic activity in different modes, we have chosen episodes in which students have made use of several modes of communication. These selected extracts have been transcribed using a multi-modal approach that records:

- speech

- the current contents of the MaLT logo editor, highlighting changes to these
- the current contents of the MaLT turtle scene
- gestures
- concurrent paper-and-pencil production

The multi-modal transcription is a pre-requisite for the analytic process, in order to identify the ways in which the meaning potentials of the various modes are exploited and combined.

Having constructed the transcription, this new representation of the data was annotated with strands of analysis related in the first place a priori to the research questions and then to themes arising from the data itself.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

As in the familiar tool (MoPiX), our analysis does not attend separately to each concern but, by examining patterns and connections across strands, addresses relationships between them. Our focus on semiotic resources brings us to identify in the analysis not only how the representations provided by the DDA are used within MaLT but also how they are taken up in verbal language and how they relate to other representations available in the context of situation.

With respect to the characteristics of the DDA, our analysis is guided by the following top concerns:

- concerns about the ways mathematical objects and their interaction are represented
- concerns about the ways representations can be acted on
- concerns about interactions between different representation systems: within the DDA; and with institutional or cultural systems of representation

The relationship between the forms of representation, the meaning potentials of these forms and the ways they are exploited and combined is a major focus of our analysis. Our social semiotic theoretical framework orients us to consider the meaning potential of the semiotic systems in use within the context of situation and to interpret the texts produced by students (in any mode), taking into account both the context of situation and the broader context of culture. The multi-modal transcriptions allow us to make direct links between the various representational systems and to track interactions among them.

With respect to educational goals, our analysis is guided by the following top concerns:

- epistemological concerns
- semiotic concerns

One strand of our analysis has been concerned with the ways in which angle has been represented by teachers and students through their use of the various semiotic systems. This strand attempts to identify different aspects of the concept of angle that are in use and the ways in which the various semiotic systems do or do not support them.

With respect to modalities of use, our analysis is guided by the following top concerns:

c.3 concerns about semiotic issues

c.5 concerns about social organization and interactions

The social organisation of the classroom and the broader institutional context have a major influence on the ways in which students may make sense of their experiences with using the DDA. One strand of our analysis has been to examine the way in which the semiotic reources and the pedagogic strategies employed by teachers and researchers in the classroom have influenced the students' own semiotic activity and their opportunities for making mathematical meanings.

A.5.8 Analysis of ITD TE with Alnuset

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The educational goals envisaged when we designed Alnuset and listed in our TE Portrait were the following:

- Exploration of properties of numerical sets;
- Exploration of what an expression denotes;
- Use of letters to generalize and prove properties. Construction of the meanings of variable, parameter and unknown;
- Manipulation of algebraic expressions and equations;
- Approximate and exact solution of equations and inequations.

Not all these educational goals have been developed in our TE. In particular, we haven't treated the following: Use of letters to generalize and prove properties and the Construction of meanings of parameter and unknown; Approximate and exact solution inequations; Approximate solution of equations.

The educational goals envisaged in our TE focus on the development of an operative and semantic control over algebraic expressions and propositions. In the following we describe our educational goals:

- Learning to practice the control of what variables and algebraic expressions indicate in an indeterminate way within a numeric domain using the quantitative method;
- Learning to practice the control of the relationship between two expressions using quantitative methods to distinguish among equivalent expressions, opposite expressions and reciprocal expressions;
- Learning to practice the control of the relationship between two expressions using formal methods to distinguish among equivalent expressions, opposite expressions and reciprocal expressions;
- Constructing a meaning for the notion of roots of polynomial and understanding the link between the roots and the polynomial factorisation;

- Constructing a meaning for the notions of equation, truth value of equation and truth set of an equation, equivalent equations, conditioned equality (an equality that is conditioned by the value assumed by the variable in the two expressions that are compared by means of the equal sign), identity.

The results of our TE have demonstrated that it is possible to achieve these didactical goals exploiting the mediation of Alnuset and in particular of the integrated use of the Algebraic Line and of the Algebraic manipulator. The educational goals have been considered as achieved when students showed to be able:

- to use the ways in which the expressions and propositions are represented in Alnuset to solve the proposed tasks, demonstrating to control the expressions and the propositions on the operative and semantic level;
- to justify the contradictions emerged in the activity making reference to the representative events mediated by Alnuset;
- to use correctly the terminology introduced by the teacher to indicate specific algebraic notions both on the protocols and in the dialogue with other participants in the teaching learning activity.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

The hypothesis presented in our TE Portrait was that the envisaged educational goals could be achieved through the integrated use of the two components of Alnuset: the Algebraic line and the Algebraic Manipulator. As a matter of fact, Algebraic Line component is oriented to the development of an algebra of quantities while Algebraic Manipulator component deals with an algebra of operations. Thus, our idea is that the integration of an algebra of quantities with an algebra of operations is crucial to develop a genuine algebraic knowledge in students (see paragraph “Common research question”, Re-CRQ3 of this document).

The TE has demonstrated that educational practices mediated by the integrated use of Algebraic Line component and of the Algebraic manipulator component are useful to develop a semantic and operative control of algebraic variables and expressions, of propositions, and to construct meaning for the equation solution.

We will give evidence of these results in the following sessions of this document, in particular in the answer to our Re-CRQ 3.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

1. How can the algebraic representations mediated by Alnuset be acted on? Which kind of relationship do these algebraic representations establish with the referential algebraic objects and phenomena?

2. Can an integrated use of these algebraic representations be effective to mediate the development of educational practices based on integration between an algebra of

quantities and an algebra of operations?
3. Are these educational practices mediated by Alnuset useful to construct meanings for the use of letters in algebra and to understand what algebraic expressions denote? Are these educational practices mediated by Alnuset useful to construct meaning for equations?

Pursuing these three research concerns we intend to frame and to validate the DF of Alnuset. In fact, each of them refers to an item of the DF notion and more precisely:

- The first research question is aimed to frame and to validate the Alnuset features from a semiotic perspective, namely, from the perspective of the mediation that it provides to control the signs of the algebraic activity (variables, expressions and propositions) at symbolic level;
- The second research question is aimed to frame and to validate the modality of use of Alnuset from a didactical and epistemological perspective;
- The third research question is aimed to frame and to validate the Alnuset effectiveness at curricular level, namely, its effectiveness to mediate the development of curricular didactical objectives on the basis of the adopted modality of use.

2 ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

Elementary algebra is the knowledge domain of reference for the PP we have designed and experimented.

Algebraic variables, expressions and propositions are the typology of signs used in this knowledge domain.

Generally speaking, students show many difficulties to control these signs at a symbolic level, namely they show difficulties to use them demonstrating to master that:

- An algebraic variable defined over a numerical set indicates the elements of that set in general terms;
- A literal algebraic expression, namely a writing composed by numbers and/or letters joined by the operation signs, indicates, in an indeterminate way, the result obtained through the sequence of operations;
- An algebraic proposition, namely a writing composed by two expressions joined through the signs of comparison “=,<,>”, indicates a truth value (true or false) that can be conditioned by the numerical values of the variables involved in its structure. The set of values that determines the truth of a proposition is the truth set of the proposition;
- It is possible to act on expressions and propositions by means of rules of computation and of symbolic manipulation that refer to conventions and properties of the operations and of equalities and inequalities. The application of these rules preserves the equivalence of expressions and propositions through the transformation and it allows the subject to demonstrate more complex rules of transformation.

Our hypothesis is that the features of Alnuset can mediate the development of the mastery of these capabilities and knowledge.

To demonstrate this hypothesis first of all we have to analyse how Alnuset allows the student to act on expressions or propositions and how it highlights the relationship between these signs and their referential objects. The adopted theoretical frame for this analysis is the Peirce' semiotic.

2. How can variables, expressions and propositions mediated by Alnuset be acted on?

To understand how Alnuset can mediate the development of control over expressions and propositions it is necessary to illustrate how expressions and propositions are represented in Alnuset and how it is possible to act on these representations.

- The algebraic variable is represented in AL by a mobile point constructed on the numbers line that can be labelled with a letter and dragged with the mouse along the line.
- The algebraic expressions are represented in AL by points on the line labelled with their symbolic form. The drag of a variable point on the line determines automatically and dynamically the refreshment of the points on the line corresponding to the expressions containing such a variable.
- Propositions are represented in a specific window of the algebraic line environment of Alnuset. A specific function allows the user to edit the truth set of this proposition. Using this function the user can construct the truth set by means of a graphical editor that the computer translates automatically and dynamically into the formal set notation.
- Expressions and propositions can be symbolically manipulated within the Algebraic Manipulator (AM) environment. The user selects the part of expression or proposition to be transformed and successively the rule of transformation he/she intends to apply on the performed selection. The rules of algebraic transformation available with the interface correspond to rules and properties of the basis of operations (addition, multiplication and power). The computer activates the rules of the interface that can be applied on the selected part of expression and it applies the re-writing rule according to the rule selected by the user, transforming the expression or the proposition at hand.

Which kind of relationship do these algebraic representations establish with the referential algebraic objects and phenomena?

Before answering this question, it is necessary to present some elements of the Peirce's semiotic frame that are useful to explain the nature of mediation that Alnuset provides to control the relationships of variables, expressions, and propositions with their referential objects.

Peirce distinguishes among three kinds of signs, namely indices, icons and symbols, according to the relationship that a sign establishes with its referential object.

We observe that in the Peirce's framework "a Symbol is a sign which refers to the Object that it denotes by virtue of a law, usually an association of general ideas, which operate to cause the Symbol to be interpreted as referring to that Object"(Peirce, 2003 - CP 2.249)

Moreover he observes that behind a rule or a convention of a sign there are always indexical and iconic links with the referential object and with its properties that can emerge through its interpretants. The mind has to learn to grasp these links to practice a symbolic control on the sign, namely behind a rule or a convention characterizing a sign it has to recognize a connection between the sign and its referential object or its properties.

When the mind grasps a factual connection between the sign and its referential object it uses the rule or the convention as index of the object indicated by that sign. An index is a sign that refers to the object it denotes on the basis of the fact that it is really determined by that object, in the sense that the sign establishes a direct, physical and pointing connection with the object .

When the mind uses the qualities or the relationships characterizing a sign to infer on properties and relationships of the referential object it grasps the iconic connection between the sign and its referential object. For Pierce an icon is a sign that has a relationship of similarity with its referential object in the sense that the sign and the object have a common quality or a common structure

Grasping the indexical or iconic connections behind the rule of Algebra is crucial to practice a control over expressions and propositions at symbolic level. Alnuset can have a very important role of semiotic mediation to develop this capability.

Let us consider some characteristics of Alnuset

We note that in Alnuset:

a) Variable as a mobile point on the line and expression as a point on the line which depends on the value assumed by the variable, highlight an indexical relationship with their referential objects (numbers on the line) through the drag of the variable point.

The presence of two expressions in a post-it associated to a point on the line may mean:

- A relationship of equality, if taking place at least for one value of the variable during its drag along the line.
- A relationship of equivalence, if taking place for all the values assumed by the variable when it is dragged along the line.
- A relationship of equivalence with restrictions, if taking place for every value of the variable when it is dragged along the line, but for one or more values, for which one of the two expressions disappears from the post-it and from the line.

The way expressions are represented on the AL of Alnuset can mediate the development of the semantic control over the numerical conditions that determine the equality between two expressions or their equivalence.

b) A proposition within the AL environment of Alnuset has an indexical relationship with its truth value that emerges through the drag of the variable on the line. As a matter of fact, the truth value of the proposition determines the colour of a marker associated to the proposition (green means true, red means false) during the drag of the variable on the line.

The numeric set represented in a formal set notation in a window of the AL has an indexical relationship with its referential object (numeric elements of that set) that emerges through the drag of the variable on the line. In fact, belonging or not of the numeric value of the variable on the line, to the formal set notation determines the colour of a marker associated to it (green means belonging , red means no belonging) during the drag of the variable.

We note that the accordance between the colour of the proposition-marker and the colour of the set-marker is a representative event that can be exploited to validate the constructed set as a truth set of the proposition.

The way propositions and numerical sets are represented can mediate:

- *the development of the semantic control over the numerical conditions that determine the truth of an equality or the equivalence between two equalities;*
- *the construction of ideas for the hidden universal and existential quantifiers required to master the truth set of a proposition.*

c) Expressions and propositions in the Algebraic Manipulator

- In the AM environment between the selection of a part of expression or proposition by the user and the activation by the computer of the general rules that can be applied on that selection (among those presented in the interface) there is an iconic relationship, namely a structural similarity of form.
- The result of the transformation can be automatically represented in the algebraic line environment to verify the preservation of the numerical equivalence through the transformation
- The available rules are open-ended, in the sense that a new rule can be automatically created once it has been demonstrated

The way expressions and propositions are manipulated in the AM of Alnuset can mediate:

- *the development of an operative control as to how to use the rules of algebraic transformation*
- *the development of semantic control as to what is preserved through their transformation*
- *the development of a theoretical control as to how to justify a new algebraic rule of transformation*

The TE has confirmed that the development of these characteristics of Alnuset are of great importance to control expressions and propositions at a symbolic level because they can mediate the development of an operative, semantic and theoretical control over them.

In this section we report two short examples that give evidence of it. This task has been solved by the students in the first activity of our TE.

> *"Consider the following assertion: The two expressions* **$–x$** *and* **$–x^2$** *considered in the rational numbers set always represent a negative number. What do you think about this statement? Justify your answer.*
>
> *Construct the two expressions on the AL and verify your answer. Then try to justify it using what is displayed on the AL during the interaction. Is there any difference among the following: $-x^2$ and $(-x)^2$ and $-(x)^2$? Use the bi-dimensional editor of ALNUSET to represent these expressions on the AL and verify your answer".*

Most students answered that "*$-x^2$ is an always positive number because the even power of a negative number is positive*". In this answer there are two errors: the first one is that $–x$ is considered as a negative number, the second one is that the power is interpreted as if it were $(-x)^2$.

Then these students represented the expression $–x$ and $-x^2$ on the algebraic line of Alnuset, dragged the variable x and observed that the point corresponding to $-x^2$ on the algebraic line is always located on negative numbers while the point corresponding to –x is positive when x is negative and vice versa

“We have verified with Alnuset that what we have written is false, so the assertion reported in the text that -x^2 is always negative is true”.

“With Alnuset we have verified that -x^2 is a negative number, $(-x)^2$ is a positive number and $-(-x)^2$ is a negative number coincident with -x^2”

Some students were quite amazed by these results

A pair of students wrote: “*-x^2 and $(-x)^2$are the same thing because making the square you always obtain a positive number…*” and after the verification with Alnuset “*…Ah, hence they are not the same thing, because in one expression the minus sign is inside the parenthesis while in the other it is not”.*

The features of Alnuset have been exploited both to destabilize students’ wrong conceptions regarding the connection of the algebraic rules used in a sign and its referential object and to develop new appropriate conceptions of this connection.

“Through the observation of the line it emerges that (except point 0) x and –x are opposite on the line. Moreover -x^2 and $(-x)^2$ are not the same thing : –$x^2$is equal to -$(x)^2$”

Let us consider this other task performed in the first activities of the experimentation.

Task:

*Write on the paper what the expression 3*x+1 represents considering x as natural number. Write an equivalent expression and use the AL and the AM to verify their equivalence.*

A pair of students write on their common paper: “*The expression 3*x+1 represents the triple of x+1”*

After this answer and coherently with their interpretation, they produce 3*(x+1) as equivalent expression of the former one.

Then, they represent the two expressions on the algebraic line and verify that they are not equivalent because they do not refer to the same point on the line and do not belong to the same post-it.

The emerging of a contradiction between the hypothesis and results by means of Alnuset helps them to reflect on the rules that characterize the two expressions and put them in connection with their referential objects.

*“Using Alnuset we have seen that the triple of x+1 is (x+1)*3 while 3x+1 is 3 times a natural number plus 1”*

Successively they conjecture 2x*x +1 to be equivalent to the expression3*x+1. They verify with Alnuset that also this hypothesis is wrong, they produce 2x+x+1 as equivalent expression that successively transform into (2+1)*x+1 and verify that these are equivalent to the given expression.

The two reported examples evidence the mediating role of Alnuset in the appropriation of the algebraic symbolism through the comprehension of how the algebraic rules characterizing an expression determine its referential object.

2. *Can an integrated use of these algebraic representations be effective to mediate the development of educational practices based on integration between an algebra of quantities and an algebra of operations?*

With Algebra of quantities we mean an algebra where the attention is focused on numerical values indicated in an indeterminate way by a variable or by a literal expression and on numerical values which determine the truth of a proposition.

With Algebra of Operations we mean an algebra where the attention is focused on the operations properties that preserve equivalence of expressions and propositions through algebraic transformation.

The design of our PP was based on the following idea: *the integration of an algebra of quantities with an algebra of operations is crucial to develop a genuine algebraic knowledge in students and Alnuset can be effectively used to support the integration of these two kinds of algebra in the didactical practice.*

The integration of an algebra of quantities with an algebra of the operations can be justified from both a didactical and an historical/epistemological perspective.

The educational research has highlighted two specific extreme students' behaviours while they operate with algebraic expressions and propositions.

There are students that Sfard and Linchevski (1992) define as pseudo-formalist, who operate with expressions and propositions in formal way, only at a syntactic level, without being able to control what is preserved through the algebraic transformation. They are unable to imagine entities (numbers, functions, truth values) that are the referents of these signs. They consider literal expressions and propositions as "things" in themselves that do not stand for anything else

In contrast, other students defined by Harper (1987) as syncopated, are unable to use the algebraic language and the methods of algebra to face the assigned tasks. They prefer to elaborate solution strategies using verbal language and the quantitative methods of arithmetic.

In both cases there is no development of a genuine algebraic thought. Both students' behaviours reveal a bad relationship with the algebraic knowledge that can be explained in terms of lack of an operative, semantic and theoretical control over expressions and propositions and on their transformation.

These behaviours can be also explained on the basis of an historical and epistemological explanation. From an historical and epistemological point of view, we note that the development of algebra as formal science and as autonomous discipline separated from arithmetics has required a lot of time. In fact, for more than two centuries Algebra has been considered as universal arithmetics. In 1830, Peacock distinguished between arithmetical and symbolic algebra. According to Peacock in arithmetical algebra attention is focused on quantities denoted by literal symbols and expressions while in symbolic algebra there is a separation of symbols from what they denote, and the attention addresses the properties of operations involved in symbols manipulation. The release of Algebra from the quantitative references of arithmetics has been a great conquest. Symbolic algebra, without these references, is the result of an abstraction process that has been worked out by mathematicians who were already able to practice a semantic control over algebraic expressions and propositions, and, if necessary, could continue to practice it.

Many students have difficulties in practicing this control because teachers often show the final point of this abstraction process, exposing them only to an algebra of formal operations. As consequences many of them consider the algebraic language as an empty language, without any meaning and any object of reference. In this context we think that the development of an algebra of quantity and its integration with an algebra of formal operations can be crucial for the didactical perspective.

These considerations are at the basis of the modality of use of Alnuset that is centred on the exploitation of its two environments, namely the Algebraic line environment and the Algebraic manipulator environment. They provide innovative operative and representative possibilities to integrate concretely an algebra of quantities with an algebra of operations in the didactical practice.

The results carried out by the analysis of the TE have confirmed the usefulness of this modality of use of Alnuset. The following section gives evidence of our TE through some examples.

3. Are these Alnuset- mediated educational practices useful to construct meanings for the use of letters in algebra and to understand what algebraic expressions denote?

Some results obtained by our TE show that the integrated use of AL and AM of Alnuset helps build educational practices which develop a semantic and operative control over algebraic variables and expressions.

Let's see some examples.

Example 1: Denotation of algebraic expressions

A didactical goal of our TE was to recognize what an expression indicates in an indeterminate way and that this represents a functional relationship that links the value of an expression to the value of its variable.

Using Alnuset we pursue the achievement of this educational goal exploiting the drag of the variable point on the algebraic line. The student can control this function mobilizing his/her own spatial, visual and motor experience.

At the beginning of the experiment students were not familiar with the movement of the variable on the line, probably due to a static vision of the numbers line that limited its possible dynamic use through the algebraic line of Alnuset. Nevertheless, results obtained by the TE clearly show the great potentiality of the drag function to develop a semantic control over algebraic expressions. In fact, once students overcame the obstacle related to a static conception of the numbers line through information and hints by the teacher, they learned easily that what an expression denotes depends on its structure and on the value of the variable. Moreover students began to use this knowledge to anticipate what an expression indicates in an indeterminate way and to use Alnuset to validate their hypothesis.

For example, at the beginning of the experiment, when students were asked to write what expressions 2x and 2x+1 denote in the Natural numbers set, they were unable to give a complete answer. In general, they only wrote that the two expressions represent natural numbers. Successively, operating in AL, they could insert the variable and the expressions correctly. However, most students didn't move the variable even if the task required it explicitly. The teacher suggested to move the variable on the line but some students tried to drag the point associated to the expressions 2x and 2x+1 instead of the variable x.

Hence the dynamic use of the mobile point associated to variables on the algebraic line has been a discovery for students as well as the dependence of the movement of the point associated to an expression through the drag of the variable.

Very soon all students could move variables on the algebraic line to explore what an expression indicates and this exploration has been crucial to construct an idea of the functional relationship that connects variables and expressions.

Let's consider the following task assigned to students in the third experimentation activity.

*a. A typical mistake students make in the approach to algebra is to state that 2*x+3 and 5*x have the same result. Use the AL of ALNUSET to show that this statement is false and to highlight that 5*x is the result of 2*x+3*x, inserting the mobile point x on the line and representing the three expressions.*

Consider the following statements:

*i) 2*x+3*x is equal to 5*x for each value of x, then the two expressions 2*x+3*x and 5*x are equivalent.*

*ii) 2*x+3 is not equivalent to 5*x but it is equal to 5*x when x is 1.*

*iii) 2*x+3*x is equivalent to 5*x because the two expressions can be transformed one into the other, while the expressions 2*x+3 and 5*x are not equivalent because they cannot be transformed one into the other.*

<u>Are these statements correct or not? Why? Verify your answers using AL and the Algebraic Manipulator (AM).</u> Discuss with the class whether these three statements are true or false, using AL and AM to verify the different opinions. What conclusions can you make about equivalence and equality between expressions?

All students could manage the expressions reported in the task on the AL. Dragging variable x they could verify that the two expressions 5x and 2x+3x denote the same quantity for each value of the variable x, while 2x+3 denotes the same quantity of 5x only for x=1.

The exploration realized with the drag of the variable has been very important for their learning.

Consider the following dialogue between a student and the teacher:

T: what do you think about the three expressions?

M: I think that they are equivalent because they are contained in the same post-it

T: But x is equal to 1

M: I know, but before moving the variable x, I try to give an answer.

T: Oh ok, but do you think that moving the variable the three expressions are still contained in the same post-it?

M: Yes, I think so. Now I move x to verify the answer. (*She moves the variable x on the line*). No they are not equivalent!!!!

In the protocol she wrote: "*We thought that 2x+3, 2x+3x and 5x moved on the line were contained in the same post-it, but it was not true. As a matter of fact, 5x is the result of 2x+3x and not the result of 2x+3. They are all equivalent when x is 1*"

The comparison between the analysis of results obtained at the beginning of TE and of those at the third lesson reveal a meaningful improvement in students' answers and behaviour regarding the use of variables and expressions on the line.

The algebraic line of Alnuset has been very fruitful because the dynamical management of the variable on the line has allowed students to assign meanings to the use of letters in algebra and to investigate what an expression indicates in an indeterminate way. By moving the variable x on the line students could easily control the values of an expression or could compare the value of more expressions, according to specific designed educational practices as described in the following sections.

Example 2: equivalent expressions

The didactical goal of the above task addresses the semantic control over expressions with the aim to recognize their equivalence or their conditioned equality

Through the quantitative approach mediated by the algebraic line, students can discover that two expressions are characterized by a relationship of:

- equivalence, when the two expressions make reference to the same point on the line and they belong to the same post-it for all the values of the variable when it is dragged along the line;
- conditioned equality, when the two expressions make reference to the same point on the line and they belong to the same post-it only for some values of the variable when it is dragged along the line.

Through the algebra of formal operations mediated by the AM students can experience that two expressions, A and B, are equivalent when it is possible to demonstrate that they have a common form, through their algebraic transformation by means of rules available via the interface. Moreover, students can verify the preservation of the equivalence through the transformation at a quantitative level, representing the transformed forms of the expression on the algebraic line and observing that they correspond to the same point and they belong to the same post-it on the line.

Before using Alnuset several students thought that the three expressions of the task were equivalent and that on the algebraic line they always have to refer to the same value dragging the variable along the line. An extract of dialogue between a couple of students and the teacher is reported in the following:

S: I think that the three expressions belong to the same post it because they are particularly similar… apart from 5x that doesn't belong to the same post-it. Let's verify on Alnuset.

Students build the three expressions on the algebraic line of Alnuset and they drag the variable point x.

S: they are not equal!

T: do you know why?

S: perhaps because… 5x is really the result of 2x+3x while it is not the result of 2x+3

T: why not?

S: because we cannot sum them!

After they have built the three expressions on the algebraic line, two different points are visualized on the line: one of them corresponds to the expressions 2x+3x and 5x and the second one corresponds to the expression 2x+3. Moreover these students visualized that :

-the expressions 2x+3x and 5x belong to the same post-it dragging on the line the point corresponding to the variable;

-the expression 2x+3 belongs to the post-it of 2x+3x and 5x when x=1.

This control on numerical quantities indicated by the three literal expressions allowed students to practice a semantic control on the notion of "equivalent expressions".

An example of answer is the following:

i) *"yes, on the algebraic line the two expressions* [2x+3x and 5x] *are equivalent"*

ii) *"yes, they are all equivalent when x=1*[2x+3x, 5x and 2x+3]*"*

iii) *"the first one* [2x+3x] *and the second one* [5x] *are equivalent since for all values of x, they are equivalent. While, 2x+3 and 5x are not equivalent because if x assumes a value different from 1, they correspond to a different value"*

In order to demonstrate the equivalence between the expressions 2x+3x and 5x students have used the available rules on the AM interface. They have transformed the expression 2x+3x into (2+3)*x , using the distributive property of multiplication over the addition, and then into 5x proving that the two expressions 2x+3x and 5x are equivalent. Some students wrote "*because they can be transformed one into the other"* or others *"the answer to questions i) and ii) is confirmed by the use of AL. The third answer iii) is confirmed by the use of AM. We have inserted 2x+3x on AM editor and we have applied the distributive property to obtain (2+3)*x; then, simplifying the numerical expression, we obtain 5x. Thus, the two expressions can be transformed one into the other. With the expression 2x+3 we cannot apply the distributive property, for this reason it cannot be transformed into 5x"*

This activity allowed students to develop a meaning for the notion of equivalent expressions by means of the integration of an algebra of quantities and an algebra of formal operations mediated by the two component of Alnuset: "*inserting these two expressions* (2x+3x and 5x, namely the first and the last form of an algebraic transformation in AM) *on the algebraic line of Alnuset, they are coincident in the same point . We have demonstrated it in the algebraic manipulator exploiting the rules* […]."

Example 3: Opposite expressions

The semantic control over expressions in order to recognize when they are opposite and to demonstrate their relationship, was a TE activity goal.

Through a quantitative approach mediated by the algebraic line, students can discover that two expressions are opposite when:

- Their respective points on the line are always symmetric in relation with point 0, when the variable they depend on is dragged on the line.

For example, a specific task required students to find the opposite expression of 5x-1 and verify their answer in the AL. Some wrote the expression –5x-1 as opposite expression. When they inserted it on the line they observed that it was not symmetric to expression 5x-1. This contradiction allowed them to understand the mistake and to find the correct opposite expression.

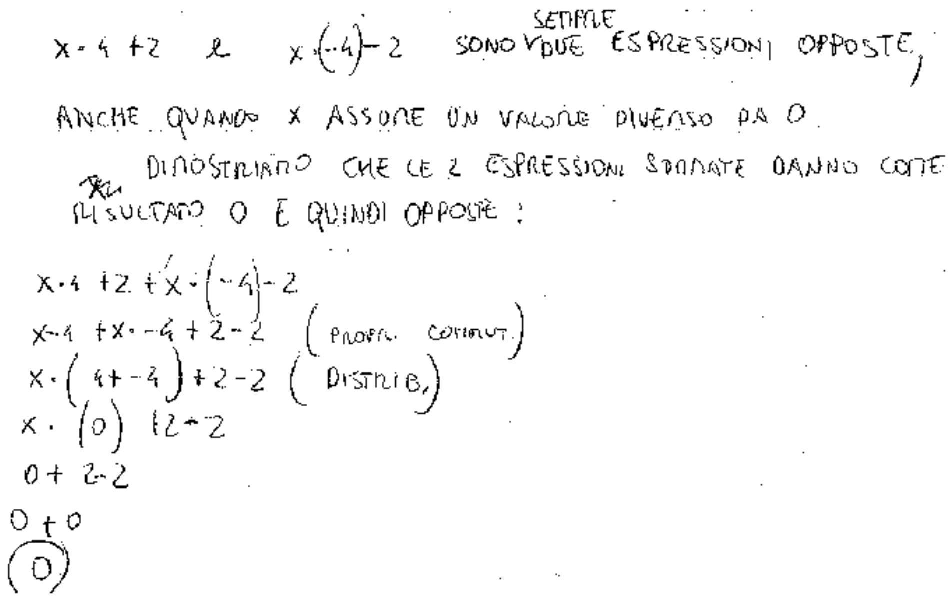
o The point on the line corresponding to the sum of the two expressions is always 0, when the variable they depend on is dragged on the line.
For example, in the previous task some students used the rule A+B=0 to prove that expressions 5x-1 and –5x+1 are opposite. Two students had written –5x-1 as opposite expression and they used the rule A+B=0 to verify that the answer was not correct. In their protocol they wrote: "*-5x-1 is the opposite function. Now we verify with Alnuset. Alnuset states that our hypothesis is not correct. As a matter of fact the result is not 0. The addition of two opposite numbers has to make 0*"

Through the algebra of formal operations mediated by the AM, students were able to experience that two expressions (A and B) are opposite when it is possible to demonstrate that A+B is equivalent to 0 or when A=-B

Once these rules were observed on the line, students were able to prove in the AM that two expressions A and B are opposite if they satisfy these rules.

In general, most students preferred to demonstrate that A and B are opposite expressions if their sum is 0. Thus they inserted the addition of the two expressions in AM and through transformation they verified that the result was equal to 0.

Observe the following protocol in which students describe steps in AM to prove that x*4+2 and x*(-4) –2 are opposite expressions.


x·4 +2 e x·(-4)-2 SONO SEMPRE DUE ESPRESSIONI OPPOSTE,
ANCHE QUANDO X ASSUME UN VALORE DIVERSO DA 0.
DIMOSTRIAMO CHE LE 2 ESPRESSIONI SOMMATE DANNO COME RISULTATO 0 E QUINDI OPPOSTE:
x·4 +2 + x·(-4)-2
x·4 +x·-4 + 2-2 (PROPR. COMMUT.)
x·(4+-4)+2-2 (DISTRIB.)
x·(0) +2-2
0+ 2-2
0+0
0

Also note that students spontaneously write the properties used in the manipulation even if this was not required by the task.

Are these educational practices mediated by Alnuset useful to construct meaning for equations?

In this section we present some results obtained by our TE showing in which way the integration of AL and AM of Alnuset helped build educational practices able to develop a semantic and operative control of propositions, and to construct meaning for the equation solution.

The following Card of our PP illustrates the sequences of tasks to realise these educational goals.

Card 8

a) *Consider the following two polynomials: x^2+2 ; $2*x+3$*
Explain what is the meaning of the equal sign between the two expression, or, in other words, how you interpret the following writing $x^2+2=2*x+3$
*What do you expect to find if you represent the two expressions x^2+2 and 2*x+3 on the AL:*
– the two points corresponding the two expression are coincident, whatever the value of x is.
– the two points corresponding to the two expressions are coincident only for some specific values of x
– other (specify)
Justify your answer
Represent the two polynomials on the AL. Drag the point x and verify your answer. Was you answer correct or wrong? Why?

b) *What does it mean, in our opinion, solve the equation $x^2+2=2*x+3$?*

Use the two-dimensional editor to built the equation $x^2+2=2*x+3$ in the Sets Space. Send the equation $x^2+2=2*x+3$ in AM, select the equation and use the rule $A \leq B \Leftrightarrow A-B \leq 0$. Translate the result produced by this rule into natural language. Transform the equation in canonical form.

c) *Make a hypothesis on the relationship among these three polynomials: x^2+2; 2*x+3; x^2-2*x-1 imagining what you could observe if you represented them on the AL and if you dragged x from which they depend on. From AM send the polynomial x^2-2*x-1to AL and use the AL commands to verify your hypothesis.*

d) *From AM send the equation x^2-2*x-1=0 to AL. Use the function E=0 to find the values of x which make the polynomial x^2-2*x-1 null. Use the function "Edit Set" to define the set of values which make the equality x^2-2*x-1=0 true.*

e) *In AM use the rule "Factor roots" to factorise the polynomial x^2-2*x-1 and find the solution of the equation x^2-2*x-1=0 in formal form.*

f) *Explain what is the relation between the solution of the equation x^2-2*x-1=0 and of the equation $x^2+2=2*x+3$. Without using the AL, write what you think about this statement: " for the values of x as solutions of the equation, the expressions belonging to the two terms of the equality in AM, when they are represented in AL belong to the same post-it.". Use AL and AM to verify and justify your answers.*

g) *Write the meaning of solve an equation and of solution of an equation.*

The tasks of this card are meant to develop well established meaning from the epistemological standpoint and concern the notions of:

- Conditioned equality
- Solution of equation
- Equivalent equations
- Truth value of an equation
- Truth set of an equation

The development of meanings for these notions is crucial to practice a semantic control of the proposition $x^2+2=2*x+3$. For this reason we have designed the tasks exploiting the integrated use of AL and AM in order to practice the integration of an algebra of quantities with an algebra of operations.

Task a) aims to explicit students' conceptions and conjectures about the meaning of equation. The students' conjectures about what is the meaning of the equal sign between two polynomial expressions are:

- Results: *"to put the equal sign between two polynomial expressions means that these expressions have the same result"*;
  "to put the equal sign between two polynomial expressions means that there exists a value of x for which the two expressions are equal";
  "the equal sign means that substituting a value of x with one of the expressions and calculating the corresponding result, we can obtain the same result substituting the same value into the other expression"

  "The two expressions"x^2+2 and $2x+3$are not equivalent because the results are different for each substitution of x"

  "$x^2+2=2x+3$ means that they are two expressions with the same result"
- equality or equivalence: *"in our opinion, two expressions are equivalent if their result is the same for all the value of x. After discussion, we think that these two expressions with the sign "=" between them don't form an equivalence but an equality"*
  "The equal sign between two expressions means that they are equivalent"

  "$x^2+2=2x+3$means that the two polynomials are one the equivalent of the other"
- equation: *"If we put the equal sign between the expressions this becomes an equation"; "by inserting the equal sign between these two expressions we obtain an equation"*

Task b) requires the algebraic manipulation of the equation $x^2+2=2x+3$. Moreover, the task requires students to make explicit their conceptions about what they means by solving an equation. The main common conjectures about this topic were the following:

"solve the equation $x^2+2=2x+3$means demonstrate the quality"; "solve the equation means to find the value of x that if substituted [into the expressions of the equation] *we obtain an equality. If we insert the equation in AM and we apply the rule, we obtain the canonical form $x^2-2*x-1=0$ "; "solve the equation means to find the value of x for which the equality is positive. The rule has subtracted $2*x+3$ to both the members of the equation. We obtain $x^2-2*x-1=0$";" solve the equation means to obtain the same values on the left and on the right of the equal sign. the rule A=B <=> A-B=0.menas that if two expressions are equal, their subtraction gives as result zero:* $x^2+2-(2x+3)=0$, using expand [computational rule of AM] we obtain the canonical form of the equation: *$x^2-2*x-1=0$ "*

"the sum of the first member with the opposite of the second one is equal to zero. We obtain the canonical form $x^2-2*x-1=0$"

In order to introduce the idea of equivalent equation and truth value, the task c) requires to make a hypothesis about the relation among the polynomials x^2+2; $2*x+3$; $x^2-2*x-1$. In the following some of the students' hypotheses:

"*the first two polynomials are equivalent only for some of the values of x. Instead, the third one isn't equivalent to the others. Moreover, on Algebraic Line of Alnuset, we have verified that when the first two polynomials correspond to the same point, the third polynomial is equal to zero"*

"In our opinion the three expressions could be equivalent. We have verified our hypothesis on AL and when the first two expressions are approximately on the same point, the third is zero…"

"We have observed that the equation $x^2-2*x-1=0$ *is the canonical form of the equation* $x^2+2=2*x+3$. *Thus, we think that if we put these expressions on AL and if we give to x the value for which the expression* $x^2-2*x-1$ *is zero, the other expressions, that is* x^2+2 *and* $2*x+3$*, will be superimposed to the same point and they will be equal."*

The construction of the truth set of the equation $2x+3=x^2+2$ and of $x^2-2x-1=0$ and the comparison of the marks' colours associated to the equations and the corresponding truth sets, allowed most students to develop the meaning for the equivalent equations

Teacher: we can move x on the line… look, here this equation is true and the other one too! So we have green balls…

I: yes, I see the green balls

F: we have to write that the two equations are true for the same values of x

I: yes, they are true for the same truth set

F: the expression on one side of the sign "=" and that on the other side of the equal sign, belong to the same post-it only for those values of x.

I: yes, for the values of x which are solutions to the equations.

To built the truth set of an equation, task d) requires to use the function E=0 available on AL which allows to find the polynomial roots. Thus, all students are able to find the roots

$1+\sqrt{2}$ and $1-\sqrt{2}$

of the polynomial $x^2-2*x-1$.

In task f) we introduce the equivalent equations notion as equations having the same truth set. In the following some Students' answers.

"*as we have said in c), for the values of x which are the solutions of the first equation* [$x^2-2*x-1=0$] *, that is the set of values which make the equality true, the other two expressions correspond to the same point and they belong to the same post-it"*

"when x is on the solution of the equation $x^2-2*x-1=0$ *the expression* $x^2-2*x-1$*is equivalent to zero and* x^2+2 *and* $2*x+3$ *are equivalent"*

"Because $x^2-2*x-1=0$ *and* $x^2+2= 2*x+3$ *are the same expression, the values of x which make true the expressions are the same. In d) we have found that the truth set of* $x^2-2*x-1=0$ *is 1+sqr2 and 1-sqr2, so these values make the equation* $x^2+2= 2*x+3$ *true too. In AL, when x assumes the values 1+sqr2 or 1-seqr2, the terms of the first equation belong to the same post-it (that is* $x^2-2*x-1$*and 0)and the terms* x^2+2 *and* $2*x+3$ *belong to the some post-it too. Thus, we can say that our hypotheses are true."*

Finally, task g) requires to formulate the meaning of solution of an equation, truth value of an equation and truth set of an equation, in natural language.

"Solve an equation means finding a value of x which makes the equation true, that is, the two terms are equivalent. The solution of the equation is the set of values which make the equality true"

"solve an equation means finding the values of x which make the equality true"

3 SPECIFY:

- THE KIND OF DATA YOU ANALYSED;
- THE SPECIFIC ELEMENTS OF OBSERVATION.

We have designed many tasks of our PP requesting students to express on paper their spontaneous considerations and hypothesis on the task at hand and on its solution before solving it with Alnuset. We think that the comparison between the initial and spontaneous idea of task solution and the solution performed with Alnuset can be useful:

- To the student, to reflect on contradictions emerged in the activity
- To the teacher, to orientate the management of the activity
- To the researcher, to study the role of the activity mediated by Alnuset in the construction of students' capabilities and knowledge

Moreover, in our TE we have given great importance to the dialogue of students working in pairs on the same computer and to the dialogue between teacher and students in specific and crucial moments of the TE

As a consequence of these choices, the DATA that have been collected are:

- Protocols provided by students
- Transcriptions of recorded dialogues between two students while solving the tasks (two dialogues recorded for each session).
- Log book. Students have been observed during the experiment, information has been collected on their behaviour and their interaction with Alnuset and the development of discussions in class have been recorded.

4 DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ.

IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

Activity Theory is the theoretical framework of reference used to study the teaching and learning activity mediated by Alnuset and to analyze the role of this artefact in a didactical perspective.

Assuming this theoretical framework as a reference, the elements of observation that we consider useful to analyse the relationship between the student (subject) and the participant in the activity (community) with respect to the didactical objective of the teaching and learning activity are:

- Contradictions emerged in the teaching and learning activity. For example, the contradiction between the initial hypothesis of task solution and the result obtained, exploiting the mediation of Alnuset and its feedbacks

- Students' interpretations of the representative events mediated by Alnuset in relation to the mathematical phenomena that have determined those events
- How the representative events of Alnuset have supported the introduction of specific notions by the teacher and the use of terms linked to those notions by participants.
- How participants have used representative events mediated by Alnuset in the dialogue among them
- How the operative functions of Alnuset have mediated :
    - the arising of objectives for the task at hand
    - the construction of meanings for specific algebraic techniques
    - the semantic control over algebraic expressions and propositions

Within the Activity Theory framework, these elements of observation are useful to understand more clearly

- the mediating role of Alnuset in the relationship between the subject and the task to be solved
- how its use has affected the other two mediating entities of the activity, namely rules and distribution of activities

5 MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

This is the list of research concerns we consider important to our work.

Characteristics of the DDA

a.1 concerns about the ways mathematical objects and their interaction are represented: here we refer to the answers to question 1. which dealt with our Re-CRQs.

a.3 concerns about the ways representations can be acted on: we refer here to answers to question 1. which dealt with our Re-CRQs.

a.5 concerns about interactions between different representation systems (if possible select among the following)

a.5.1 within the DDA: The interactions between algebraic line representations and Algebraic manipulator representations. As a matter of fact, the expressions represented in AL can be sent to AM by means of the function "send to manipulator"; vice-versa, the expression represented in AM can be send to AL by means of the "send to line" function.

a.5.3 with institutional or cultural systems of representation the tasks of our PP are designed in order to produce contradictions between solutions built with paper and pen and solutions built with Alnuset components. Learning occurs overcoming these contradictions.

b) Educational goals

b.1 epistemological concerns

b.2 semiotic concerns

In particular, we refer to the Re-CRQ 2 answer in which we explain in details how the semiotic, epistemological and symbolic concerns of our PP allowed students to reach the envisaged educational goals exploiting the didactical functionalities of Alnuset (i.e. exploiting some characteristics of Alnuset and their modalities of employment defined in the designed tasks). We refer to the session "Validation of the DDAs and PPs" of this document for a detailed list of the envisaged educational goals

c) Modalities of use

c.1 concerns about the tasks and their temporal organization. Our PP is organised by two main items: Algebraic and Polynomial Expressions and Equations. Each of them is organised in sub items. The item "Algebraic and Polynomial Expressions" is organised in: Exploring what an expression denotes through an algebra of quantities, Exploring equivalent expressions integrating an algebra of operations with an algebra of quantities, Exploring opposite and reciprocal expressions, Exploring roots of polynomials.

The item "Equations" is organised in: Exploring equations as conditioned equality between two expressions, Exploring particular kinds of equations.

Each item is dealt with a card composed by a sequence of tasks.

Each session of our TE is devoted to one card (it happened that during some sessions we have dealt with more than one card, for example we have solved some tasks of the successive card)

c.2 concerns about the functions to be given to the DDA and their possible changes

c.3 concerns about semiotic issues the tasks of our PP are designed with the aim to achieve the semantic control of expressions and propositions in algebra.

c.4 concerns about the relationship between knowledge referred to DDA functioning and knowledge referred to the educational goals. We refer here to the answers to question 1. which dealt with our Re-CRQs.

c.5 concerns about social organization and interactions.

Our PP is organised in order to develop the willingness and capacity to: work collaboratively; participate effectively in class discussion; question one's own work through critical evaluation of the work of others. In particular, we refer to Activity Theory which is used to frame the pedagogical strategy used in our PP, to analyse the contradictions that can emerge in the development of the didactical activities and the mediating role of Alnuset representations to overcome them.

c.6 institutional and cultural concerns. The contents addressed in the PP are part of the Italian maths curriculum for the first and second year of upper secondary school.

Up to now we have considered only concerns regarding the characteristics of the DDA and the educational goals and not all those regarding the modalities of use.

A.5.9 Analysis of ITD TE with Aplusix

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The main educational goal specified in our TE portrait was to understand the structure of numerical expressions. In particular:

- Learn how to represent a numerical expression as a tree.
- Learn how to "build" a tree given a numerical expression or an expression described in natural language.
- Learn how to "read" an expression represented by a tree.
- Learn that there is only a linear expression for a tree while there could be different tree representations for an expression represented in linear form

In general, the analysis of TE results shows that it is important to distinguish between a procedural idea of structure of an expression and a conceptual idea of it. In particular, we refer to Kieran work (1989) on the construction of a structural idea of an expression in elementary algebra. The procedural view of the structure is what Kieran calls surface structure and refers to the arrangement of the different terms and operations that make up an algebraic (arithmetic) expression. In other words, the surface structure is the "order" in which symbols appear in the expression. Kieran defines the conceptual view of the structure as systemic structure. Systemic structure refers to the properties of operations within an algebraic expression and to the relationships between the terms of the expression that come from within the mathematical system. For example the expression 2*(3+5) can be written either as (3+5)*2 (using the commutative law) or 6+10 (using the distributive law).

Analysing students results in the TE we can state that the specified educational goals were achieved by students if we consider a procedural idea of structure. At the end of the experiment, the comparison between the final test results and those of the initial test, shows that most students have acquired competencies in translating a natural language expression into a linear expression and vice-versa. Moreover, they can also convert a linear expression into a tree representation. Nevertheless, some mistakes concerning the use of parentheses and the priority of some operations over others still emerge in students' protocols, in particular, in tasks requiring to convert the tree representation into a linear representation. As highlighted in the following (see p. 7), competencies involved in converting tree representations into linear expressions probably differ from those involved in converting a linear expression into a tree representation. In particular, procedural competencies prove insufficient to perform the former task. Structural aspects of numerical expressions need to be managed when transforming a tree into a linear expression.

For these reasons, results analysis shows that, with respect to the envisaged educational goals, even if students can express the superficial structure of the expression, they do not quite manage to fully comprehend the systemic structure of a numerical expression.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

Our hypothesis specified in the TE portrait was that the envisaged educational goals could be achieved in educational practice through the use of Aplusix and in particular of the tree-representation mode feature (Controlled tree representation and Mixed tree representation).

As we have specified in the a-priori analysis, our TE was based on the Semiotic registers of representation (R. Duval) and on the Activity theory frame. Our hypothesis was that specific tasks related to an integrated use of Aplusix tree representations, could favour students in understanding the structure of numerical expressions.

We can confirm that Aplusix was an important tool in accomplishing this educational goal. The tree-representation mode feature has been used to build meaningful activities to allow students to learn how to represent a numerical expression as a tree and vice-versa.

As shown in the following section, students master some structural aspects of linear expressions. Remarkable improvements could be noted by comparing results of the initial test with those of the final one.

Moreover, the feedback characteristic provided by the system was useful to design activities based on the Activity theory frame. According to it, learning can emerge overcoming contradictions that can appear during educational activities. Tasks of our TE were designed to be a source of contradiction through a comparison between pen and paper solution and the solution performed in Aplusix. These contradictions emerged during the experiment when the student compared his answer written in a paper and pen environment with that provided by the system. The system did not say where the mistake could be, but it allowed students to know if the solution was correct or not. Moreover, through appropriate and well designed activities, the teacher could provide hints to help them find a correct solution.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

Is the educational activity based on the conversion among different representations (linear representation, tree representation, natural language) effective to mediate the understanding of the structure of numerical expressions? In particular,

1. Is this conversion among representations effective to teach how to represent a numerical expression as a tree?
2. Is this conversion effective to teach how to “build” a tree given a numerical expression or an expression described in natural language?
3. Is this conversion effective to teach how to “read” an expression represented by a tree?
4. Is this conversion effective to teach that there is only a linear expression for a tree while there could be different tree representations for an expression represented in linear form?

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

At the beginning of the experiment, students’ competencies on numerical expressions were mainly of the computational type - students knew the priority of operations and the use of parentheses and were able to use them in a procedural way to calculate the results of simple numerical expressions.

When they had to convert an expression in verbal language, they realized a stenographic translation of its linear representation. Their conversions never reflected structural aspects of expressions. These observations, among others, clearly showed their difficulty to control the expression on a structural level. Students could manage the “superficial” structure (Kieran, 1989) of expressions, but they could not manage their “systemic” structure (Kieran, 1989).

Thus, the **main educational goal** of our PP was to understand the structure of numerical expressions in a "systemic" way.

Our hypothesis was that specific tasks related to an integrated use of Aplusix tree representations, would be able to help students understand the structure of numerical expressions. The theory of Semiotic registers of representation (R. Duval) states that the ability to represent a given mathematical concept in at least two registers and to perform conversions from one register to another could be an indicator of conceptual understanding of a notion. In our PP we have considered three registers of representation for an expression: linear representation, tree representation and natural language.

Aplusix is a tool that allows us to compare and validate linear representation and tree representation for expression. The modalities of employment of this tool inside the PP are based on the Activity theory frame. According to it, learning can emerge overcoming contradictions that can appear during educational activities. Tasks of our PP are designed to be a source of contradiction through a comparison of the pen and paper solution and the solution performed in Aplusix.

In the following we present some specific examples trying to answer our common research questions.

1. *Is this conversion among representations effective to teach how to represent a numerical expression as a tree?*
2. *Is this conversion effective to teach how to "build" a tree given a numerical expression or an expression described in natural language?*
3. *Is this conversion effective to teach how to "read" an expression represented by a tree?*

The three reported questions can be connected to these three **specific educational goals**: to learn the tree representation of linear expression, to learn to convert a linear expression (expressed in symbols or in natural language) in a tree representation, and to learn to convert a tree representation in linear expression (expressed in symbols or in natural language).

The main **characteristic of Aplusix** that we have exploited to pursue these educational goals is the feedback is available with this artefact to verify the equivalence between two linear expressions, between two tree representations or between linear expression and tree representation.

Moreover, we have used two specific characteristics of the system to edit expressions as trees: the Controlled Tree representation and the Mixed Tree representation.

When a tree is edited in the Controlled Tree representation there are some constraints and scaffolding - internal nodes must be operators and leaves must be numbers or variables.

When a tree is edited in the Mixed representation there are less constraints: each leaf of the tree can also be an usual representation. The usual representation can be expanded as a tree by clicking the "+" button that appears when the mouse cursor is near a node; a tree, or a part of a tree, can be collapsed into a usual representation by clicking the "-" button that appears when the mouse cursor is near a node.

Let's explain in more details the **modalities of employment** of these characteristics.

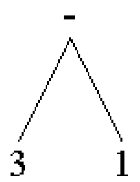
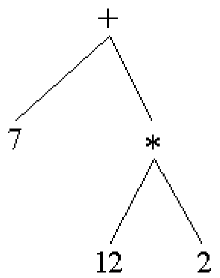
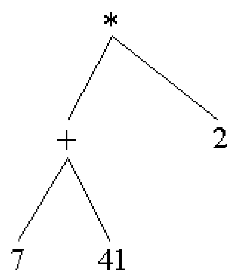
The Controlled tree representation was mainly used at the beginning of the experiment to teach students to build a tree representation. During the experiment it was abandoned while the Mixed representation was used along the whole experiment.

This choice is justified by the fact that our activities were in general oriented to use the system to validate solutions previously constructed using paper and pen. In this context the Mixed tree representation was the most appropriate form of editing for this aim because the student was free to insert in leaves not only numbers but also expressions.

The Mixed tree representation was used as a validation tool: working with paper and pencil, students were asked to construct tree representation by linear expression or linear expression by tree representation. They could verify their answers using the Mixed representation mode of Aplusix inserting the expression and building the tree representation. They could verify if the tree representation of the screen and the tree representation performed with paper and pen were coincident or not.

Some typical tasks proposed to students are reported:

"Write the linear expression for each tree representation. Then, verify your answer in Aplusix"

1.		
	…	…
5.		
6.		
	…	…

"Write a tree representation of the following linear expressions in the right side of the table. Then verify your answer in Aplusix"

1.	$3+4\times5$	
2.	$3\times4+5$	
3.	$3\times(4+5)$	
4.	$3+(4\times5)$	
5.	$\frac{2}{5}+\frac{3}{2}+\frac{1}{7}$	
6.	$\frac{2+5}{3+2}$	
7.	$\frac{2+5}{6}\times\frac{2+1}{7}$	

The analysis of students' solutions has highlighted that, opposite to our expectations, the second task was easier with respect to the first one. Difficulties emerged in the second task are mainly depending on the poor experience of students in tree construction.

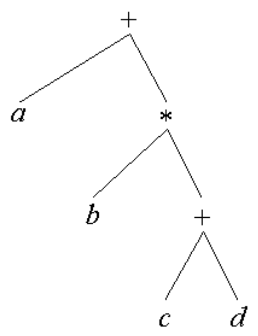
On the contrary, students' solutions to the first task contained many mistakes that were not present in those of the second task. They depend on the use of parentheses - many students wrote the linear expression without using them, even when they were required.

This could be due to the fact that, in order to translate a tree representation into a linear expression the student has to decide whether to insert parentheses or not, while when he has to construct a tree starting from a linear expression he has to translate parentheses but not to insert them in the tree.

A more detailed analysis highlights that to accomplish the second task the student has to know the syntactical structure of the tree (how to build a tree) and has to respect some computational rules. The student has to build the tree taking into account that collapsing bottom-up the tree he has to find the sequence of computation described by the linear expression. This task strengthens procedural skills, or, in other words, the "superficial structure" of a numerical expression.

On the contrary, to accomplish the first task procedural skills are not sufficient. The student has to interpret the tree structure.

Consider the following tree:



If the student reads the tree in a procedural way, he could fail in choosing between these three expressions: a+b*(c+d) or a+(b*(c+d)) or a+b*c+d. In order to convert the tree in linear form inserting parentheses in the correct place, it is important to read the tree interpreting its systemic structure and this entails the capability to manage the numerical expressions in a structural way. For this reason, a-posteriori we think that perhaps it would have been more appropriate to propose to students the second task before the first one.

The last card proposed another task centred on the conversion of an expression into the three representations (linear, tree and natural language). This is the text of the task:

"Construct the linear expression that corresponds to each statement. Then, construct the tree of each expression and verify your answers in Aplusix (inserting the linear expression and building the tree representation)"

Addition between 2 and 7		
Subtraction between 53 and 35		
Product among 3, 4 and the addition between 5 and 2		
Difference between 15 and the quotient between 9 and 3		
Quotient between 6 and the quotient between 4 and 2		
…	…	…

Most students performed this task in a correct way. At this point, quite interestingly, students "felt" obliged to use Aplusix but they preferred to compare their answers with schoolmates. If answers were different they preferred to collaborate rather then to operate with Aplusix to understand which of them was the correct one and why.

This behaviour could be due to the fact that, up to that moment, Aplusix was mainly used to build tree representations by linear expressions and vice-versa, with the aim to verify if the two representations were equivalent. It was not used to verify if the statement expressed in natural language was equivalent to the linear expression or to the tree representation. Thus, students needed to discuss with their peers to validate their answers before the scheduled common discussion.

During the development of the experiment we have observed an improvement in students solutions. In particular, during class discussion their language in converting expressions or tree representations was more appropriate with respect to the language used in the initial test.

In this latter, when converting a linear expression into verbal language students realized a stenographic translation of its linear structure using a lot of words and long statements.

At the end of the experiment, once they had learned tree representations for linear expressions, they began to manage the “systemic” structure of an expression and not only the “superficial” structure and this was reflected in the way they used the verbal language, the linear and the tree representation in the proposed conversion tasks.

4. *Is this conversion effective to teach that there is only a linear expression for a tree while there could be different tree representations for an expression represented in linear form?*

The **specific educational goal** is to learn that given a tree representation it is possible to convert it into a linear expression while given a linear expressions it is possible to build more than one equivalent tree representation of it.

The **characteristics** of Aplusix used to answer this question are:

- The feedback to verify the equivalence between two linear expressions, between two tree representations or between linear expression and tree representation.
- The Mixed tree representation to edit expressions as trees.

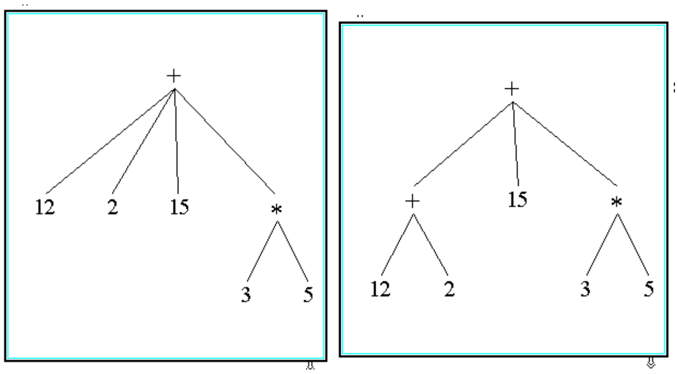
The **modality of use** of these characteristics is different with respect to those previously reported.

Let’s consider this task.

“Consider the expression 12+2+15+(3*5). Build a tree representation in Aplusix. Is it the only possible representation corresponding to the linear expression?”

Initially all students constructed only a tree representation for the expression. Through discussion and cooperation more representations emerged. The key point of the activity has been the justification of the equivalence among the different tree representations realized.

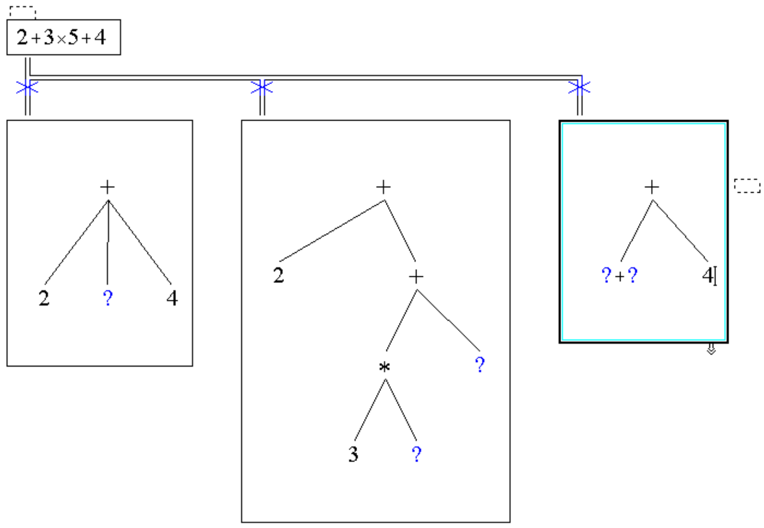
To justify their equivalence some students made reference to the properties of operations. For example, observing the figure below, some students noted that it is possible to construct different tree representations because addition is characterised by the commutative property.

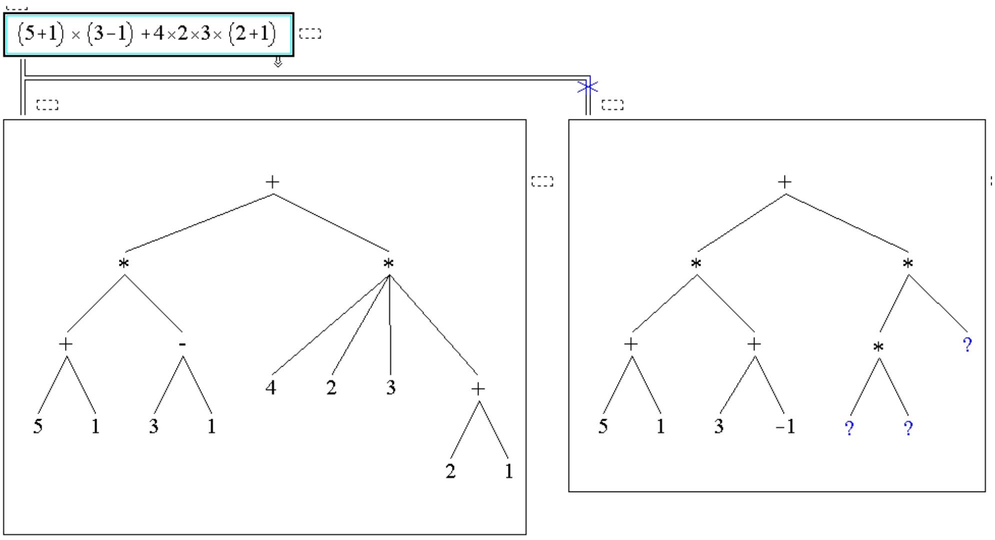


Differently from previous tasks, this one orients students to reflect on the priority of operations and on their properties and to focus on the fact that different forms can have a common structure that can emerge through the use of those properties.

Another task that has produced interesting didactical results in learning, was the following:

"Complete the tree representations in the following diagrams and verify the answers given in Aplusix."





This task is quite unusual in the experience with Aplusix. We think that it is an interesting task because students have to focus on structural aspects of a numeric expression. Students have to interpret the representations assigned in the task and to compare them. Through their comparison students receive hints on structural aspects of the numerical expressions needed to replace the question marks. In this solution the feedback is crucial.

Opposite to our expectations, a lot of students performed this activity successfully.

3. SPECIFY: - THE KIND OF DATA YOU ANALYSED; - THE SPECIFIC ELEMENTS OF OBSERVATION.

Collected data were mainly constituted by:

- A logbook for each session containing the observed behaviour of students, their interaction with the system and some short dialogues between students, between students and researchers and between students and teacher. This logbook was constructed by researchers who were present during the experiment;
- Answers provided by students in their copies;
- Two transcriptions of dialogues between two students during each session.

The analysis of the experiment is based on these data.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ. IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

To support our answers to the Re-CRQ we have analysed students answers (students' copies and transcription recording) to detect contradictions with their previous answers.
As a matter of fact Aplusix was used according to the following pedagogical strategy:

Task solution based on the use of pen & paper vs. task solution based on use of the tool. This strategy could be a source of contradictions, and overcoming it could be the motor of learning.
Activity Theory model allows us to explain and to model the learning process mediated by this pedagogical strategy. In particular, we have analysed the role of Aplusix in helping students overcome contradictions. We have also analysed discussions and interactions among students because, in some cases, they could be necessary to account for contradictions and overcome them.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

The following concerns guided our analysis

Characteristics of the DDA

a.1 concerns about the ways mathematical objects and their interaction are represented

a.3 concerns about the ways representations can be acted on

As previously described, we have mainly used the feedback of Aplusix to verify the equivalence between two linear expressions, between two tree representations or between linear expression and tree representation. This procedure allowed students to work alone and validate their answers using Aplusix whenever they were unsure about their answers. Moreover, we have used two specific characteristics of the system to edit expressions as trees - the Controlled Tree representation and the Mixed Tree representation.

In our analysis these characteristics have been used .

a.5 concerns about interactions between different representation systems (if possible select among the following)

a.5.1 within the DDA

The interaction between the tree representation modality and the linear representation modality in Aplusix is important for our analysis because it provides two registers of representation (Duval)

a.5.3 with institutional or cultural systems of representation

As we have previously described, we have analysed the comparison between two systems of representations: Aplusix and paper and pencil environment. The comparison between paper and pencil solutions with those mediated by Aplusix led to the emergence of contradictions. We have analysed these contradictions and the role of the system in helping student to overcome them

b) Educational goals

b.1 epistemological concerns

The main educational goal specified in our TE portrait was to understand the structure of numerical expressions. We have used an epistemological analysis to define the specific goals of the educational activity. In the section“Validation of the DDAs and PPs” of this document we show the detailed list of the envisaged educational goals and we explain in which way they have been achieved or not achieved.

b.3 cognitive concerns

We have also accomplished an analysis based on the role of Aplusix to favour specific cognitive processes involved in attaining the educational goal. In the section“Validation of the DDAs and PPs” of this document we explain in which way this analysis was accomplished.

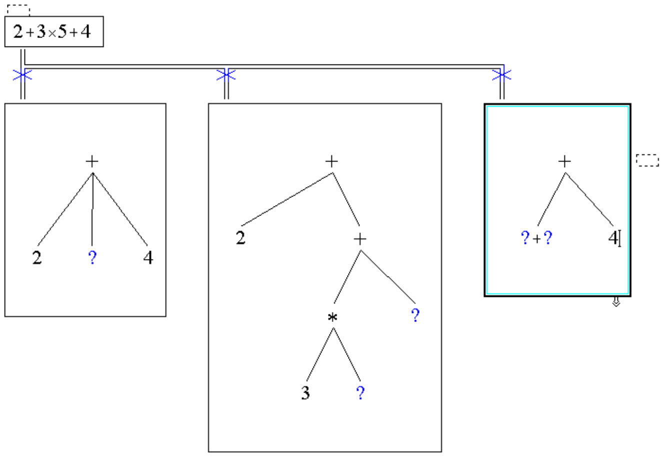
c) Modalities of use

c.1 concerns about the tasks and their temporal organization

The analysis highlighted that the order of tasks proposed in our PP should perhaps be changed. We now see that students ought to have the opportunity to build tree representations of numerical expressions before being called on to build numerical expressions from tree representations, as was the case in this experiment. As a matter of fact, we have observed that the competencies involved in converting tree representations into linear expressions differ significantly from those involved in converting a linear expression into a tree representation. In particular, procedural competencies proved insufficient to perform the former task.

c.2 concerns about functions to be given to DDA and their possible changes

In some cases, the modalities of use of specific characteristics of Aplusix are very different with respect to the modalities probably planned by the designer. In particular, we refer to a specific task requiring to build different tree representations of a given numerical expression (see the figure below). This kind of task is unusual. Nevertheless, in our opinion it is cognitively richer than the others because it obliges students to consider the structure of the expression. However, Aplusix provides students with a strong support for this task and, opposite to our expectations, few difficulties emerged in solving it.



c.5 concerns about social organization and interactions

In the TE we have analysed students behaviours in developing the willingness and capacity to work collaboratively, in participating effectively in class discussion to question one's own work through critical evaluation of the work of others. In particular, we refer to the Activity Theory framework used to frame pedagogical strategies used in our PP, to analyse the contradictions that can emerge working with the paper and pen environment and with the system.

c.6 institutional and cultural concerns

The contents addressed in the PP are part of the Italian maths curriculum for lower secondary school.

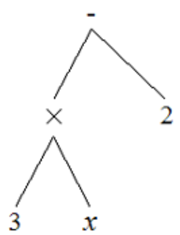
A.5.10 Analysis of MeTAH TE with Aplusix

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

In one class we have analyzed two types of observable:

- Students' responses to exercises on algebraic transformations obtained in a written test. There has been an evolution of students4 performances in the tasks of algebraic transformation.
- Students' responses in conversion activities between the tree register (RT) and the natural language register (RNL), e.g.,



« différence du produit de 3 par x et de 2 »

Another element, which was not included in the design of the experiment, was used to attest that the educational goals were achieved in this class. It was the fact that the teacher used the tree register during a remedial lesson with only one group of students. During this lesson, the teacher treated students' errors linked to operations priority. These students showed better performances in tasks of conversion from the tree into natural language registers than students who had not benefited from this remedial lesson

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

Based on the results of our experimentation, we are convinced about the soundness of the hypotheses we set up for the teaching experiment. However we wish to mention two issues that may explain why some of the educational goals were not, or were only partially achieved. The first issue is the time factor. Two teachers asked to shorten the scenario for institutional reasons explained elsewhere. In addition, in two experiments, the teachers could not implement the post-test provided by the scenario. Another problem is the constraint related to the curriculum. For teachers, our scenario is not consistent with the objectives of the curriculum.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

How do the new semiotic register “tree representation” and its articulation with the usual and the natural language registers help the students understand the structure of algebraic expressions, in the sense specified below?

Educational goals of our experiment are the following:

The students will be able to (curricular goals):

- identify the form of an algebraic expression given in either of the following representation systems: tree, natural language, symbolic language;
- convert an algebraic expression given in one representation system into another one;
- solve problems involving algebraic expressions given either in natural language or in symbolic language (usual representation).

We think that the achievement of the above mentioned educational goals will indicate that the students:

- understand the structure of algebraic expressions;
- are able to distinguish between procedural and structural aspects of algebraic expressions.

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

The same pedagogical plan has been enacted in three different classes (1 Grade 9 class and 2 Grade 10 classes – let us note the classes C1, C2 and C3 respectively) with three different teachers.

The activities proposed in our pedagogical plan rely on 3 semiotic registers of algebraic expressions representation: usual (RU), natural language (RNL) and tree registers (RT). The students were supposed to be familiar with the usual register they use in their math classes. By natural language register we mean the mathematical vocabulary used in algebra rather than an ordinary “wording” of algebraic objects (e.g., reading the expression 2x+y as the sum of a product of 2 by y and of y rather than two x plus y). The later is more usual in math classes, but the current curricula incite teachers to introduce and to make use of the former, mainly in the introduction to algebra. Tree representation was novel to the students.

Students were first administered a pre-test with a few traditional numerical and algebraic exercises such as calculate, factor, develop and simplify, but also a task requiring to convert algebraic expressions given in usual register into the natural language register and vice versa. The results of the pre-test, namely in the C1 and C2 classes, confirmed our hypothesis that even Grade 10 students keep having difficulties in algebra, and most of these are related to the structure of expressions, despite of the fact that in France, the most of algebra is taught in junior high school, i.e., between Grades 6 and 9.

For the C1 class, unfortunately we did not manage to gather enough information about the conditions of the teaching experiment. The geographic location of the school did not allow observations of the sessions. The results obtained indicate that the teacher implemented the scenario to fulfil her experimental contract rather than to integrate it into her pedagogical activities, contrary to the teachers of the C2 and C3 classes. These two teachers made an effort to integrate the scenario and make the best of it in order to help their students, mostly those having difficulties with algebra. The results from the pre-test show that C3 students had a rather good level in algebra and only a few difficulties were observed. Also a few difficulties were observed in the tasks of conversion RU $\rightarrow$ RNL.

The results from the C2 class are more interesting with respect to our research question. In this class, we analysed in detail the conversion tasks RU $\leftrightarrow$ RT and the treatment tasks in RT for numerical calculation.

The results obtained at the post-test in standard algebra tasks in RU show a positive impact of the teaching experiment on the students' achievements in treatment tasks in RU. The improvement concerns mostly respecting priority of operations, distributive property of multiplication and handling the minus sign. On the other hand, difficulties related to the powers have not been overcome. A possible reason for this is the fact that there were only few exercises involving powers in the scenario, thus this notion has not been worked out sufficiently.

3. SPECIFY:

   - THE KIND OF DATA YOU ANALYSED;

   - THE SPECIFIC ELEMENTS OF OBSERVATION.

The students were working either on paper worksheets or with Aplusix. In this case, all students' actions have been recorded by the system and are available for the analysis through a replay system. Moreover, in one experimental class, the introduction of the tree register with Aplusix software was done by one of the researchers who designed the pedagogical plan, and the whole session was video recorded. Thus, the kind of data we analysed is:

- students' answers to pre-test questions (Aplusix files and paper worksheets);
- students' productions corresponding to the experimental tasks (Aplusix files and paper worksheets);
- video recording of the introductory session in one class (C2) only;
- students' productions to a class exam including algebra tasks of treatment in usual register.

These data provide the following elements of observation:

- students' strategies and answers provided to the tasks, namely erroneous answers and strategies are of interest for us since they can indicate to what extent the educational goals are achieved globally and to see for each student if her/his state of knowledge has evolved or not
- the way the tree register has been introduced to the students (interactions between the teacher and the students, institutionalisation). This is a crucial moment of the scenario that can have an impact on the rest of the activities. Unfortunately, we could not gather this kind of data but in one class (C2) where the introduction to the new semiotic register of representation was introduced by one of the researchers who had designed the teaching scenario.

Note that initially, we planned to administer a post-test with similar activities as were assigned in the pre-test. Actually, the post-test was not proposed as planned due to the time constraints the teachers were facing. Instead, each teacher included a few algebra tasks of treatment in usual register similar to the first part of the pre-test (development, factoring, solving equations) into a regular class exam. We could analyse the results to these tasks and compare them with the results obtained at the pre-test. Moreover, a homework was assigned to the students at the end of the "Learning" unit with conversion tasks RT ↔ RNL and it was used to assess the students' skills.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ. IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

In accordance with the Duval's semiotic registers of representation theoretical approach, we assume that an understanding of the structural aspect of algebraic expressions is evidenced by the students' capacity to convert an expression given in one register into another one, mostly when the target register is natural language or tree register. To perform such a conversion, one needs to identify the structure of the expression. Indeed, to build a tree or to describe the expression in natural language (in the sense specified above), it is necessary to identify the operators involved in the expression and the sub-expressions that are their arguments or operands. Therefore, we analysed the students' productions with a particular attention paid to the conversion tasks. We were looking at both the results provided by the students and the procedures they used to perform the tasks.

The results at the pre-test confirmed our hypothesis that even Grade 10 students keep having difficulties in treatment tasks with algebraic expressions. In particular, errors related to the lack of understanding of the structure of expressions have been observed, such as transforming 2+3x into 5x and errors in treatment of powers and minus sign, e.g., $3(-5)^2 \rightarrow 3 \pm 25$; $(-3x)^2 \rightarrow \pm 3^2x$; $(3x)^2 \rightarrow 3x^2$.

The conversion tasks RU → RNL proposed in the form of communication games showed as well the difficulties with recognizing the structure of expressions, despite the fact that the students succeeded in the games. In fact, the expressions given in RU were described in a sort of oral register "close" to the algebraic usual register. Thus the students described actions allowing reproducing the same expression. For example, the expression $\frac{(3x+2)(3x-1)}{a-(x+2)}$ was read as : "*open a parenthesis, 3 x plus 2, close the parenthesis, open a parenthesis, 3 x minus 1, close the parenthesis, the whole over a minus open a parenthesis, x plus 2, close the parenthesis*". This kind of message is given what we can call oral register, which emphasizes procedural aspect of the expression rather than structural. In this register, the expression is read from left to right and the message contains ambiguities. However, this register makes use of implicit codes the students share (e.g., a short break is made where a parenthesis should be put), and these helped them to succeed in the task.

In the C1 class, one group of students (G1), rather low attaining, benefited from a work with Aplusix, in controlled mode, on conversion tasks RNL → RT during one 50-minute session, while the other group (G2) could not attend the session because of technical problems. The results in the conversion task RT $\rightarrow$ RNL assigned at a homework to both groups, show a significant difference between the performances of students from the two groups (see table below).

	RNL with « structural » aspect	RNL with « procedural » aspect (oral)
G1 (15 students)	10	5
G2 (15 students)	3	12

These results indicate that the students from the G1 group have grasped better the structure of expressions than the students from the G2 group. This points out the efficiency of the session on conversion tasks RNL $\rightarrow$ RT.

Furthermore, we have compared the results obtained in the pre-test with those obtained in the post-test in the C1 class. In the pre-test, 11 students out of 35 had important difficulties in treatment tasks in RU, which manifested themselves through errors in handling with powers, minus sign and priorities of operations. The post-test put forward that among these 11 students, 5 have progressed. The other 6 keep having difficulties mostly with powers.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

A) CHARACTERISTICS OF THE DDA:

A.1 concerns about the ways mathematical objects and their interaction are represented

A.3 concerns about the ways representations can be acted on

A.5 concerns about possible interactions, connections with other semiotic systems, including the representations provided by other DDAs

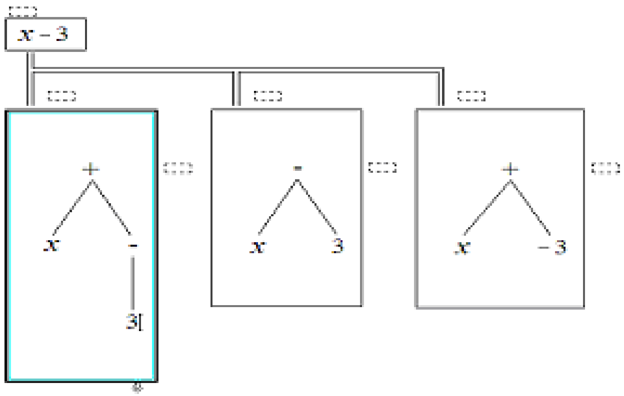
A.5.1 within the DDA

A.5.3 with institutional or cultural systems of representation

The mathematical concept at the core of our teaching experiment is algebraic expression that cannot be dealt with otherwise then by means of its representation. Since the TE involves a DDA, namely Aplusix, the concerns about the ways algebraic expressions are represented (A.1) are of the highest importance. In Aplusix, two different representations of expressions are available and can be acted on: usual and tree representations. There are several modes of interaction with Aplusix. At a more general level, there are training and test modes, which differ from each other by the feedback that is provided only in the training mode. Within the tree representation, three different modes are available: free tree mode, controlled mode and mixed mode. While the controlled and mixed modes provide some scaffolding during the tree edition, in the free mode the user is completely free to build whatever tree s/he wishes. In the TE, the choices of interaction modes were made according to the educational goals. In the analysis of TE, we have to pay attention to the ways representations can be acted on (A.3), since it turns out that the success to a given task can be explained by the fact that the interaction mode did not allowed the students to make mistakes rather than to the actual understanding of the mathematical concept at stake. For example, in the C1 class, where we noticed that the students have not reached the expected understanding of the structure of expressions, very few difficulties have been observed in the conversion tasks RNL → RA done with Aplusix in the controlled mode. Finally, in the TE, the new register of representation of algebraic expressions, tree representation, interacts with the usual representation already available in Aplusix (A.5.1) and with the natural language register, which is not taken into account in the software (A.5.3). The interactions take place through the conversion tasks.

- EDUCATIONAL GOALS:
  - B.1 epistemological concerns
  - B.2 semiotic concerns
  - B.3 cognitive concerns

As regards the educational goals, semiotic and cognitive concerns (B.2; & B.3) are at the core of the TE and they guide our analysis of the students' abilities to make conversions between the three semiotic registers of representation of algebraic expressions (Duval 1995). From the epistemological point of view (B.1), we look at the ways the specificities of each register are dealt with and are taken into account by the students. As an example, we can mention the discussion about the different meanings of the "minus" sign that took place during the introductory lesson in the C2 class:



In the expression x-3, the minus sign can be given three different meanings leading to three different trees (difference transparent in usual representation):

- "opposite" of a number, which is a unary operator (tree on the left);
- "difference", which is a binary operator (tree in the middle));
- "sign of a relative number" (tree on the right).

C) MODALITIES OF USE:

C.2 concerns about the functions to be given to the artefact and their possible evolution

C.3. concerns about semiotic issues

Besides the analysis of students' achievements with respect to the educational goals that we set up in the TE, we question the choice of tasks we proposed to the students and the interaction mode in which they had to be solved (C.2). For example, in retrospect, it turns out that the treatment tasks in RA we proposed at the end of the "Learning unit" (except of calculation with numerical expressions) are of a limited pedagogical interest and are extremely hard to perform (C.3).

A.5.11 Analysis of MeTAH TE with Alnuset

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

Let us remind that the main educational goals of the pedagogical scenario is that students construct:

3. the meaning of function as a relationship between dependent and independent variables;
4. the meaning of the notions of equation and inequations as statements that are true for some values of a variable;
5. the meaning of equivalence between expressions as statements that are true for all values of the variable;
6. the meaning of a solution of an equation as a value of the variable for which the equation is true.

So far, only the part of the experimentation related to the notion of function has been analysed. As is shown below (cf. answers to the questions about CRQ), the students were able to perceive the given functions as relationships between two variables. Here are examples of the students' answers attesting this understanding: "*x^2 moves depending on x* (in French, *x^2 bouge en fonction de x*)",

"*we cannot drag x^2 with the mouse because x^2 depends on x, therefore we have to touch x to make x^2 move*". The dynamic representation of the functional relationship in the Algebraic line component of Alnuset definitely contributed to this achievement.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

Based on the results of our experimentation, we are convinced about the soundness of the hypotheses we set up for the teaching experiment. However we wish to mention two issues that may explain why some of the educational goals were not, or were only partially achieved. The first issue is the time factor. The whole scenario was planned for only 3 hours. The analysis in terms of praxeologies available in Alnuset shows that these are quite different from the institutional ones, and making a link between those requires a longer time. Moreover, the development of some instrumented action schemes that underpin Alnuset techniques may be a rather long process, as is pointed out by the a priori analysis of the system. The second issue is the articulation between representations available in Alnuset. In studying functions, we postponed the introduction of the Cartesian plane component to the end of the sequence devoted to functions. This decision was consistent with our hypothesis that conceptualisation of the notion of function requires to be able to dissociate a function from its graphical representation. However, it turns out that articulating horizontal dynamic representation of a function with its 2D static representation may contribute to establish links between instrumental and institutional techniques (cf. studying of variations of functions).

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

Do instrumental techniques for studying functions and for solving equations and inequations available in Alnuset, which are based on visual observations of expressions (their position on the algebraic line, colour feedback…) contribute to the conceptualisation of the notion of function, equation and inequation?

2. ANSWER YOUR RE-CRQ. WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

3. SPECIFY: - THE KIND OF DATA YOU ANALYSED; - THE SPECIFIC ELEMENTS OF OBSERVATION. 4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ. IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

Notion of function

Numerous research works report about difficulties encountered by students with the notion of function. Il appears that a large part of them are due to obstacles of both epistemological and didactical nature (Chauvat 1999)[10] as will be shown in what follows.

Epistemological considerations

Historically, the notion of function took a long time to fully develop. First, the notion of curve appeared in Greek mathematics. Their notion of curve, mainly as a geometric locus, served as a tool for solving geometric problems until Descartes (17th century). Descartes starts considering a curve as a set of points characterized by a distinctive property, called symptom, which can be expressed by means of an algebraic relation. A symptom emphasizes an algebraic relationship between the involved quantities, but conceals the functional relationship of dependency between them. With Leibniz, Newton and Bernoulli, the notion of function becomes a mathematical tool, and later, with Euler in the 18th century, it becomes an object of study. New types of problems, such as searching for minimums or maximums or study of

[10] Chauvat, G. (1999), Courbes et fonctions au collège, *"Petit x"* 51, 23-44.

tangents to curves, gave rise to the development of differential and integral calculus and led to the notion of numerical function where two conceptions coexist: function=analytical expression and function=curve. As Sierpinska (1992)[11] points out, they are both obstacles to the general notion of function. Indeed, the conception function=curve (i.e., its graphical representation) does not allow to perceive a dynamic process the function models: one quantity varies in relation with variations of an other one. The perception of this relational dynamics is eclipsed by the statics of a finished drawing.

Didactical considerations

The teaching of functions in the French schools seems to reinforce this obstacle (Chauvat 1999). In fact, students first encounter curves and functions through the example of linear function linked to the study of proportional relationship between quantities (Grades 7 and 8). The general notion of function is taught later (Grade 10) and is based on the notion and use of graphical representations, which leads to assimilation by the students of a function with its curve.

Research hypothesis (which is our CRQ specified for the case of the notion of function)

Based on these considerations, we hypothesize that grasping the notion of function requires be able to:

- perceive a function as a relation of dependence between variable quantities;
- dissociate the notion of function and its graphical representation.

Thus, in the pedagogical plan, consistently with this hypothesis, functions under study are first explored in the Algebraic line component, where the dynamic relationship between a variable x and an expression dependent on x can be observed. The notions of image, pre-image, domain and co-domain of a function can be approached, as well as variations of a function can be studied. Graphical representation of the functions in Cartesian plane component is proposed later, in connexion with Algebraic line component.

Experimentation

The experiment was implemented in one Grade 10 class in a private high school. Initially, two 1- hour sessions were planned, but the second session could be extended to 2 hours. Both sessions took place in a computer lab, students working in pairs. During the first session, the students were split in two groups of 20 and 14 students respectively, the second session took place with the whole class.

The notion of function was addressed in the first session. Besides the objective of familiarisation with Alnuset, the aim was to study two functions prescribed by the French curriculum: $x \to x^2$ and $x \to 1/x$ (the class teacher required to include the study of these functions into the experimental activities). Regarding these functions, the students were first asked to observe the relationship of dependence between x and x^2 by observing that, on the one hand, when x moves on the algebraic line, x^2 (or 1/x) moves accordingly and, on the other hand, x^2 (or 1/x) cannot by dragged with the mouse. Then, their attention was drawn to the

[11] Sierpinska, A. (1992), On understanding the notion of function. In *The concept of function: Aspects of epistemology and pedagogy*, MAA Notes n°25, 25-58.

way x^2 (or 1/x) moves when x moves on the line. The aim of this task was to develop an instrumental technique allowing to determine variations of a function. This technique is based on observation of the movement of f(x) when x is dragged along the algebraic line: when x and f(x) move in the same direction, the function f is increasing, when they move in opposite directions, f is decreasing.

Below are reported the main results. The analysis is based on the data we collected, namely students' written productions, some students' discussions that were audio recorded, observers' notes.

Results

1. The dependence of x^2 on x is easily perceived by the students due to the dynamic representation of x and x^2 on the algebraic line. Follow some of the students' answers to the questions "What happens to x^2 when you drag x" and "Can you drag x^2 with the muse?":

"*x^2 moves depending on x* (in French, *x^2 bouge en fonction de x*)"

"*we cannot drag x^2 with the mouse because x^2 depends on x, therefore we have to touch x to make x^2 move*".

However, some students are not satisfied with such observation and they try to characterize the relation of dependence between x and x^2:

"*x^2 is proportional to x*"

"*x^2 is going farther*"

"*when x is on the negatives, x^2 is always positive, a square is always positive*".

2. As regards the variations of the functions, only 2 pairs of students out of 17 succeeded in this task. Most of the students failed to interpret mathematically their observations and these remained at a level of a description what they saw on the screen:

"*when x is positive, x^2 moves to the right, when it's negative, it moves to the left*"

"*x^2 goes farther and farther*"

"*x^2 never goes under zero*"

"*x^2 goes until zero, then goes to the right*"

Several hypotheses can explain these results:

- The question asked to the students was very vague. The students did not know what they were expected to observe. Thus, their answers can seem legitimate. However, the following question, which asked directly to deduce variations of the functions from their observations, indicated more clearly what kind of observations were expected.

- The notion of variation of a function was not understood by the students. Many students asked for explanations as regards this notion.

- The instrumental technique for exploring variations of a function is significantly different from the techniques students were taught. Two techniques are used to study variations of a function in a Grade 10: one is based on the "reading" of variations from the curve that is a graphical representation of the function (i.e., the function is increasing on an interval I if the curve is "going up" on I, the function is decreasing on I if the curve is "going down" on I); the other is based on comparing f(a) and f(b) given two abscissas a and b from I such that a<b: f is increasing on I when f(a)<f(b), f is decreasing on I if f(a)>f(b). The representation of a function on the algebraic line consists of a unique pair of x and f(x), which represent any pre-image and its image. This representation does not allow a direct comparison of two images of a function: one has to imagine that the movement of x generates another pair (x, f(x)), interpret the movement of x to the right in terms of increasing the value of x, observe in which direction f(x) moves and interpret the movement of f(x) in the same direction in terms of an increase and the movement in the opposite direction in terms of a decrease. At this moment, it would be appropriate to link the representation of the function on the algebraic line with its graphical representation by means of Cartesian Plane component of Alnuset, which would perhaps help the students to observe the link between a horizontal displacement of f(x) and its displacement on the curve representing the function. In the experiment, Cartesian plane was introduced later, after having worked on functional equations and inequations of the type f(x)=k, , f(x)>k, f(x)=g(x)…. This choice, although consistent with our hypothesis that in order to conceptualise the notion of function, it has to be dissociated from its graphical representation, turns out as not being well judged.

Answer to the (part of) research question (related to the notion of function)

The results described above show that the dynamic representation of expressions on the algebraic line of Alnuset can contribute to perceiving the dynamic functional relationship between two variables, which is necessary (but not sufficient) to grasp the notion of function. The experimentation shows also that the feedbacks coming from the tool are not easily interpreted in terms of mathematical properties of objects that are manipulated. This emphasizes the importance of the role of a teacher in managing students' instrumental genesis intertwined with the targeted mathematical knowledge.

Notions of equation and inequations

To be developed

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

A) CHARACTERISTICS OF THE DDA:

A.1 concerns about the ways mathematical objects and their interaction are represented

A.3 concerns about the ways representations can be acted on

A.5 concerns about possible interactions, connections with other semiotic systems, including the representations provided by other DDAs

A.5.1 within the DDA

A.5.3 with institutional or cultural systems of representation

The mathematical concepts at the core of our teaching experiment are the notions of function, equation and inequation. They cannot be, like all mathematical concepts, dealt with otherwise then by means of their representations. Since the TE involves a DDA, namely Alnuset, the concerns about the ways these notions are represented in it (A.1) are of the highest importance. In Alnuset, these notions can be approached in three different components corresponding to different registers of representations: Algebraic Line providing a dynamic representation of the relationship between two variables, Cartesian Plane allowing to visualize a graphical representation of this relationship, and Symbolic Manipulator, in which transformations on algebraic expressions are possible by means applying transformation rules on the expressions. In our TE, only the first two components are used. The ways representations can be acted on (A.3) were taken into account in the analysis of instrumental genesis in students. Thus, for example we could observe two different types of using Alnuset in solving simple equations, such as $x^2=4$: some students solved first the equation, either mentally or on paper, and then verified their solution with Alnuset. Such usage often led to an erroneous answer consisting in providing only the positive solution, which is due to the conception "$x^2=k^2 \Leftrightarrow x=k$". Other students used dragging of the variable x on the algebraic line and looked for values of x for which x^2 and 4 coincide. In most cases of such usage, the students were able to find the two solutions of the equation. Finally, in the analysis of our TE, we were looking for elements in students' productions and discussions that would evidence about articulation between different representations available in Alnuset (A.5.1), as well as articulations between other representations encountered in the math classes (A.5.3) such as table of variations of a function or table of signs used in solving inequations.

- EDUCATIONAL GOALS:

  B.2 semiotic concerns

  B.3 cognitive concerns

  B.6 cultural and institutional concerns

Our TE was built on the hypothesis that the dynamic representation of the functional relationship between variable quantities and its articulation with static graphical representation will contribute to the conceptualisation of the notions of function, equation and inequation. In analysing the outcomes of the TE, we were naturally interested in semiotic issues (B2). Also, besides analysing how the representations available in the DDA contribute to the achievement of educational goals, we tried to figure out what conceptions are developed by the students in interaction with Alnuset (B.3), and eventually what is the distance between these conceptions and the targeted knowledge (B.6).

C) MODALITIES OF USE:

C.1 concerns about the tasks and their temporal organization

C.2 concerns about the functions to be given to the artefact and their possible evolution

C.3. concerns about semiotic issues

C.6 institutional and cultural concerns

These concerns guided our critical retrospective view on the activities we proposed to the students (C.1) and of the way the different registers are articulated (C.3). For example, we were questioning the choice to postpone the introduction of graphical representation of expressions to the end of exploratory activities, thus assigning to this representation the role of a means of validation. Since the TE aimed at helping students progress in learning math notions that are at the core of Grade 10 curriculum, the institutional knowledge was taken as a reference in our analysis (C6).

A.5.12 Analysis of Unisi TE with Aplusix

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The PP proposes an introduction to structural aspects of algebraic thinking through the manipulation of numerical expressions. Our educational choices have been motivated by the epistemological assumption of not considering algebraic calculation as a generalization of arithmetical computation but as manipulations based on the equivalence. As a consequence we aimed at promoting a structural approach also in arithmetic. The possibility of identifying a structure in a numerical expression is thought being supported by exploiting the innovative representation given by the software, the tree representation (TR), which has been used together with the standard representation (SR), and the natural language (NL).

The general educational goals of the PP which has been specified in the portrait are:

3. Anticipating the introduction to the algebraic calculation, as a manipulation based on the equivalence.
4. Introducing to the *"structure sense"* [12] of an expression.

Within such global aims, we pointed out more specific educational goals, that focus on numerical expressions in the perspective of introducing algebraic calculation. They are:

1'. acquiring the notion of equivalence between expressions;

2'. acquiring the structure sense for numerical expressions.

In particular, the role played by the properties of the operations to demonstrate the equivalence between expressions is considered a key point in the delicate passage from arithmetical to algebraic computations.

We think that the educational goals we pursued have been reached by most students. In fact, as a result of our teaching experiments, we can say that most students have learnt:

- the notion of equivalence between expressions (ed. goal 1');
- to distinguish between a structural reading and a procedural reading of numerical expressions, and to recognize numerical expressions having the same structure (ed. goal 2).

To verify and document the achievement of the educational goals from a comprehensive standpoint, we set up a pre-test and final test device. In the final test, students were explicitly required to provide procedural and structural readings of expressions, and to recognize those expressions that have the same structure (ed. goal 2). For example, the following one is a question of the post-test:

Among the following expressions, given in SR and in TR, identify those that have the same structure:

[12] For *structure sense* of an algebraic expression we adopt the definition of Hoch and Dreyfus (2006):

"A student is said to display structure sense for high school algebra if s/he can:

- Recognise a familiar structure in its simplest form.
- Deal with a compound term as a single entity, and through an appropriate substitution recognise a familiar structure in a more complex form.
- Choose appropriate manipulations to make best use of a structure."

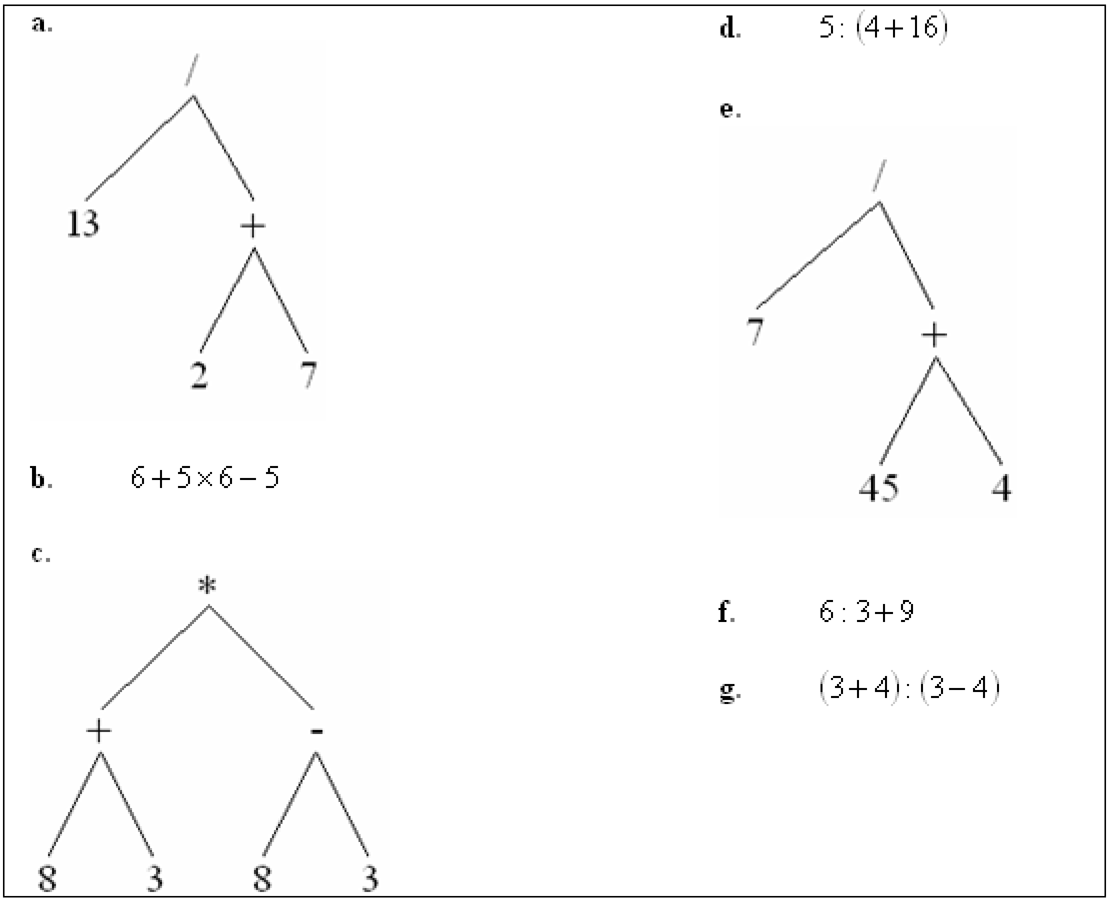


Figure 1

The task reported in Fig. 1 has the objective to verify whether students have gained competences in reflecting on structural aspects of a numerical expression both in TR, where, according to our hypothesis the structure is more evident, and in SR, where the structure remains hidden. In addition, this question also requires to be able to perform conversion, either operationally or mentally, in order to compare the given expressions.

The analysis of the protocols reveals that many students have internalized the recourse to the tree representation for comparing expression given both in SR and in TR. In the following, a protocol (Fig. 2) where a student, who correctly solves the exercise mentioned above, shows to be able to face this type of task in a structural way. It is evident how the 'empty tree' (that is a tree without operands) is more effective than putting operators without operands in SR. for what concerns putting into evidence the structure of an expression.

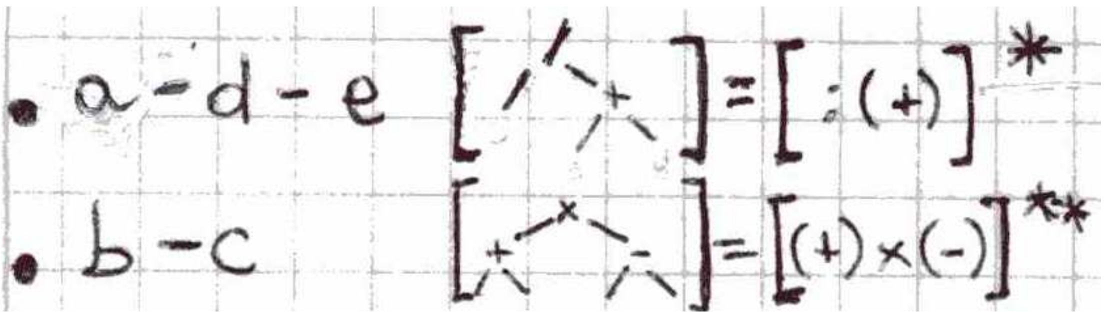


Figure 2

It seems that the role played by trees in putting into evidence the structure of a numerical expression makes it possible for students to recover to TR when they needed 'to see' the structure of an expression. It is also shown in some protocols belonging to activities performed before the post test, where students have preferred to compute some expression given in natural language (written in a structural way) by means of trees: it can be considered an evidence that the tree has become an internalized tool. According to our hypotheses this behaviour has been triggered by means of the schemes of use of the trees, and in particular of the sub-trees. In fact, when asked to compute on the trees, most students have implemented the strategy of selecting a sub-tree and substituting the result deriving from the computing to it. In this way, since the calculation is structured by the tree and, by means of actions on the tree, students have become able not only to do right calculations but especially to reflect on what they were doing.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS? EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

The PP was based on the following hypotheses that link the use of the DDA with the Educational Goals:

3. the tree representation provided by Aplusix is a vehicle for supporting the structural sense of an expression. In particular, according to our theoretical framework, TR may be exploited by the teacher as a tool of semiotic mediation for making students acquire a structural sense of expressions (ed. goal 2 and 2');
4. the presence in Aplusix of different kinds of representation systems is effective for the envisaged educational goals. The potential of treatments in tree representation outlines a crucial aspect of algebra: the sense of structure (ed. goal 2). The activities of conversions between different registers, and in particular between standard representation and natural language, have the goal to make the students conscious of a substantial difference between arithmetic and algebra (ed. goal 1).

In the portrait we hypothesized to verify the achievement of the educational goals by gradually monitoring the students’ production all along the implementation in class of the pedagogical plan.

After the experimentation our hypotheses have been validated by the obtained results.

By analyzing students' semiotic processes and productions, we managed to identify key elements that provide evidence of the role of TR in students’ learning. As expected, we found traces of students’ experience with the tree representation even when working in paper and pencil. Students seem to use TR when they are required to think in a structural way and they have become able to compare structurally an expression, that means referring to it without numbers, (see Fig. 2 in the example above) which is the basis on which starting to build the algebraic thinking.

Furthermore, we gained evidence on the exploitation of the tree as a semiotic mediator by the teacher. The following excerpt is an example, taken from the classroom discussion that concludes the activity one of the didactical cycle 2:

126. Teacher: Ok. Let’s go a bit further. We observed that among the four operations, there are two which are commutative, that means that we can invert the leaves and obtain new trees which are equivalent to the previous ones; whereas the other two operations are non-commutative since if we invert (gestures as shown in Fig. 5) the leaves.

Figure 5

126.Amalia (interrupting): They are not equivalent any more.

127.Teacher: They are not equivalent.

128.Cora: The result changes.

129.Teacher: The value which represents our number changes. Very well.

In the episode reported in the transcription, the teacher exploits the tree as a semiotic mediator tool through a combined uses of different semiotic resources:

-words

-gestures (see in Fig. 5 the gesture for indicating the inversion of leaves)

- inscriptions on the blackboard.

For further discussion and examples, see below answer to CRQ.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

How does TR (tree representation) in Aplusix can be exploited as a tool of semiotic mediation for the structure sense of a numerical expression? In particular, is it possible to identify semiotic chains going from artefact-signs to mathematical signs, through which students can acquire a mathematical meaning of structure of an expression?

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

The tree representation of Aplusix (*DF pole 1: set of features of the tool*) can be exploited by the teacher as a tool of semiotic mediation for the structure sense of expressions (*DF pole 2: specific educational goal*). In fact, the modalities of use of the software proposed in the PP (*DF pole 3: modalities of employing the tool*) have revealed that TR has been an effective artefact to reach the envisaged educational goals, and in particular the structure sense. With this respect, fundamental roles have been played by

- the individual reports, that have fostered individual reflections and the explicitation of personal meanings.
- the classroom discussions, in which the teacher could promote the evolution from artefact-signs towards mathematical signs and mathematical meanings.

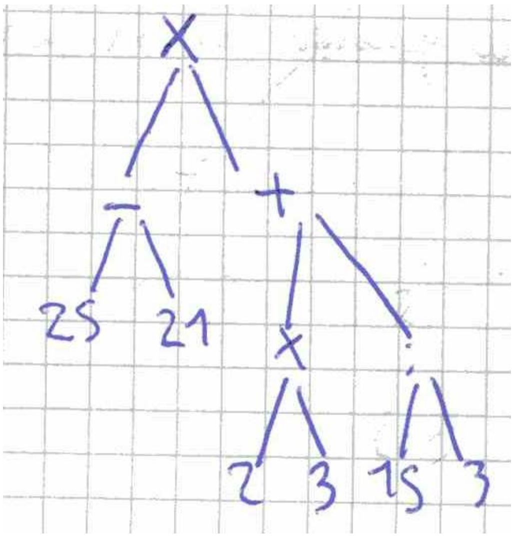


Figure 4

These teaching methodological tools (in our TF they are elements of a didactical cycle) are consistent with the perspective adopted in the MTF. In particular they face the issues that 'individuals can share *representations* only through sharing the perception of the perceivable *representing*. However, it may happen that in spite of common perception of the *representing,* they fail to share the *represented*'.

An example is provided by the following excerpt from a student's production in the post- test.

In this exercise students were asked to compute two numerical expressions, given in natural language (one is given as it is read in structural way, the other as it is read in procedural way). We have observed that a common behaviour of students in solving the exercise is that t' structural way ('the product of the difference between 25 and 21 and see Fig. 4) is first of all converted in a tree representation, and then it is calculated by means of reducing the levels of the tree.

In the frame of the Theory of Semiotic Mediation, we identified specific elements that contribute to the evolution of meanings and related signs in terms of semiotic chains starting from artefact-signs and going towards mathematical signs. Key elements of these semiotic chains are:

- artefact-signs: signs that are directly related to the use of Aplusix (e.g. talking in terms of branches, nodes, …);
- pivot signs: signs that can play key-roles in the process of evolution of meanings; generally they have both references to mathematical context and artefact context;
- hybrid signs: signs that blend features related to the artefact with typical mathematical signs;
- mathematical signs: signs that are clearly recognizable as belonging to the domain of mathematics.

An example related to the tree has been proposed above (Fig. 4). The sketch of the tree in paper and pencil can be considered an artefact-sign related to the relative specific feature of Aplusix (TR).

Another example can be found in the protocol that we have discussed above and that we report again, for the reader's sake:

3. Teacher: Ok. Let's go a bit further. We observed that among the four operations, there are two which are commutative, that means that we can invert the leaves and obtain new trees which are equivalent to the previous ones; whereas the other two operations

are non-commutative since if we invert (gestures as shown in Fig. 5) the leaves.

Figure 5

3. Amalia (interrupting): They are not equivalent any more.
3. Teacher: They are not equivalent.
3. Cora: The result changes.
3. Teacher: The value which represents our number changes. Very well.

The teacher's discourse (composed by speech and gestures) in line 126 is an example of hybrid sign:

- "commutative operations" refers to mathematical signs;
- "inverting the leaves" relate to the artefact.

Furthermore, "inverting the leaves" constitutes an example of artefact-sign.

3. SPECIFY: - THE KIND OF DATA YOU ANALYSED; - THE SPECIFIC ELEMENTS OF OBSERVATION.

Kind of data:

Qualitative: audio and video- recordings, students' worksheets and written reports, field notes by teachers and researchers.

Specific elements of observation:

Students' and teacher's words, written signs, gestures.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ. IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

To answer our Re-CRQ, we have made an integrated use of the data.

In particular, we have looked for relationships between students' and teacher's signs in respect to

- the DDA's characteristics on the one hand,
- the mathematical meanings underlying our educational goals on the other hand.

In the frame of the Theory of semiotic mediation, we have tried to describe the unfolding of the *semiotic potential* of Aplusix with respect to the notions of equivalence between expressions and in the acquisition of a structure sense for numerical expressions. In doing so, we have tried to describe those *semiotic chains, which* origin with the use of the artefact, to solve the task, and develop through signs that are more strictly related to such use (as the *artefact-signs*), to signs more recognizable in the domain of mathematics (what we call *mathematical signs*).

Another important element of analysis has been the action of the teacher in coordinating and leading the classroom discussions. With this respect, we have analysed how Aplusix can be exploited by the teacher as a *tool of semiotic mediation* for the envisaged educational goals, keeping in mind the background goal of introducing algebraic computation.

5. Make explicit which concerns guided your analysis process and how.

Our analysis process has been strongly guided by our theoretical frame, and by the need to answer the CRQ and the SRQs. Being so goal-oriented, the process has been guided by the same concerns that were at the base of the setting of the teaching experiments, and specifically:

a) Characteristics of the DDA

a.1 concerns about the ways mathematical objects and their interaction are represented

a.3 concerns about the ways representations can be acted on

The theoretical framework we refer to, which is the Semiotic Mediation, requires considering the two concerns mentioned above as the two faces of the same coin. In fact, according to the semiotic potential construct, by one hand the artefact refers to mathematics (concern a. 1); by the other hand it is a means to accomplish mathematical tasks (concern a.2). As a consequence, the analysis process aims at identifying the ways in which the artefact has worked in vehiculating the mathematical meanings it embeds, and at the same time, how it has functioned in the accomplishment of specific tasks.

b) Educational goals

b.1 epistemological concerns

This concern guides *what kind of traces* we are interested to look for. In our semiotic approach, the emergences of specific signs are considered evidences for what concern the achievement of specific educational goals. For instance the meaning which is given to the tree has an epistemological connotation.

b.2 semiotic concerns

This concern guides *where* to look for. Our analysis is performed through the analysis of the emergence of signs and their role in students' evolutions from personal meaning to mathematical meanings. For instance, the structure sense emerges from activities with the artefact in TR, then evolves through both the production of trees in paper and pencil and the production of hybrid signs in SR, which aim at putting into evidence the structure of an expression (see Fig. 2).

c) Modalities of use

c.2 concerns about the functions to be given to the DDA and their possible changes

c.3 concerns about semiotic issues

In our analysis these two concerns are considered together. We analyze the functioning of the DDA and how some possible changes are related to the emergence of specific mathematical meanings (semiotic chains).

Specific Research Questions

1. REPORT YOUR SRQ.

1. Can we say that the students have arrived to distinguish between a structural reading and a procedural reading of numerical expressions?

2. Is it possible to observe and describe the realising of the hypothesized semiotic potential of the artefact in relationships to the tasks?

3. Is it possible to observe if and how the teacher has exploited the realisation of such semiotic potential to guide students towards mathematical meanings?

4. Have any artefact-signs emerged? Which ones? When? Is it possible to identify the semiotic chains that link the emerged artefact-sign to the mathematical meanings?

These SRQs are not the ones formulated at the beginning of the PP. Their modifications have been prompted by the modifications that the PP underwent in the course of its refinement.

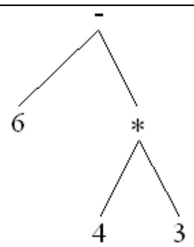
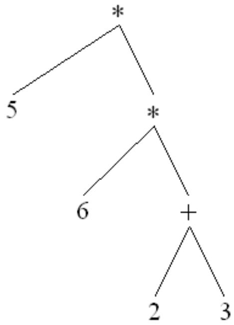
2. ANSWER YOUR SRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

a. Yes, we can say that such didactic goal has been reached. We get such information from the final test, where most students could answer the specific questions in correct way.

For example, students were asked to provide a structural reading and a procedural reading of different expressions, given in SR or TR. The table below reports the results in students' answers in one of the two classes in which the PP has been implemented:

Expressions	Students' performances					
	Procedural reading			Structural reading		
	Correct answers	Incorrect answers	Omitted	Correct answers	Incorrect answers	Omitted
$12-5\times 7$	14	3	2	11	3	5

	12	4	3	10	2	7
$(3+4):(10-4)$	17	0	2	12	0	7
$\frac{2}{3}+4\times 6$	14	0	5	11	1	7
	16	0	3	11	0	8

2. and 4. Yes, it is possible to describe the realising of the hypothesized semiotic potential of the artefact in relationship to the tasks, and it is strictly related to the emerging of artefact-signs. Some semiotic chains have been identified.

As explained answering the re-CRQ, we identified some categories that contribute to the evolution of meanings in the semiotic chain: artefact-signs, pivot signs, hybrid signs, mathematical signs. The following is an example taken from the discussion at the end of the Didactical Cycle 1, when the class is discussing about the signs that Aplusix shows during computations:

12. Amalia: Well, there are three signs…well, those two vertical lines are when the passage is right and concluded

[…]

30. Teacher: […] What does it mean "to be right"?

31. Martina: That you didn’t make any mistake in the computations

32. Amalia: That you have not mistaken anything and you can go to the following passage

33. Martina: The computations, the sign…

[…]

54. Teacher: […] And how can we that do not use the computer, understand that things are right without seeing the signs? Why are they right?

58. Ambra: Because if the computation follows a logical thread, it is right
59. Teacher: Because if the computation follows a logical thread, it is right. What does it mean to follow a logical thread?
60. Martina: To do certain operations

[…]

66. Teacher: […] Why are passages right? What does if mean to have the passages right? Where does it lead the logical thread? […]
67. Amalia: Because basically the last passage must give you the result of the first one
68. Teacher: The last passage must give you the result of the first one: what does it mean?
69. Amalia: And yes because basically if you solve the first passage the result must be…equal to the second
70. Teacher: Let's help her to tell it well

[…]

72. Ambra: Yes because finally the result is the simplification of the first, each passage has the same result
73. Teacher: and so?
74. Amalia: Basically, if we have…I don't know…6/3 and we reduce to the minimal terms it comes 2, doesn't it? (The teacher writes on the blackboard 6/3 and 2)
75. Amalia: so I tell that 2 is the result of the first passage

[…]

Figure 6

89. Teacher: […] How can we say that? […] How can we say that the result of 6/3 is 2? In mathematics, when we speak, how can we say that the result of 6/3 is 2?
90. Cora: That the result of 6 divided 3 gives 2
91. Teacher: Yes, but…what do we say of these two (pointing to 6/3 and 2 with the two hands, Fig. 6) here?
92. Valentina: That they are equivalent each other[13]
93. Teacher: That?
94. Valentina: Yes, that they are equivalent one another, they are equivalent
95. Teacher: And what does if mean that they are equivalent?
96. Amalia: That they are equal…
97. Students: That they have the same value

The teacher starts the discussion by focusing it on the interpretation of the feedback signs of Aplusix. As emerged in the written sheets, at the beginning of the discussion students' assign the meaning of "right passage" to Aplusix symbol ǁ. According to our theoretical framework, This is an artefact-sign, taking its meaning from the artefact world it is expected to develop towards a mathematical sing referring to the notion of equivalence. During the discussion we can observe the semiotic chain (a sequence of hinged signs) through which the first artefact-sign evolves through the guide of the teacher.:

right / no errors (from line 11 to line 61)

↓

(connecting) passages with the same result (from line 62 to line 80)

[13] "Si equivalgono"

↓

they are equivalent each other[14] (lines 83 and 85)

↓

they are equivalent[15] (from line 85)

The semiotic chain come into existence under the constant stimulus of the teacher who asks the students either to make explicit the meanings of the signs involved ("what does it mean", lines 30, 56, 63, 86) or to elaborate on their expressions ("Let's help her to tell it well", line 65; "How can we say that?", lines 80, 82). Note that in this elaboration different signs, be they either belonging to Aplusix, as || or to oral language, are related in a semiotic game generating a complex web of meanings.

By repeating and re-formulating students' contributions on the one hand, and making explicit reference to mathematics language on the other hand, the teacher fosters the weaving of a texture of meanings in which the meaning of equivalence comes to be sided and overlapped to that of right passage. This double interpretation of Aplusix feedback signs is the core of the semiotic potential of this specific feature of the DDA in solving the given tasks.

3. Yes, it is. For what more specifically concerns the role of the teacher, we have observed interesting didactic strategies in the classroom discussions. As we expected, we gained information from the video-recordings of the classroom discussions, which show how the teacher has lead the evolution of meanings towards the mathematical ones. Such teacher's strategies are based on a semiotic perspective that allowed the weaving of textures of meanings centered on the dialectics between artefact-signs and mathematical signs. They are still under analysis and definition, but we can roughly mention some of them:

- asking to go back to the task with the DDA (relationship with the characteristics of the DDA and the modalities of use);
- focus actions: the teacher focuses on certain aspects related to the use of the artefact;
- amplification actions: the teacher repeats some signs provided by some students (e.g. words, gestures), so to make them audible to all the students, and therefore sharable in the classroom;
- pointing towards mathematics: the teacher makes explicit reference to mathematics domain (relationship with the educational goals);
- semiotic games: the teacher repeats some signs provided by the students, and inserts them in a suitable context with respect to the mathematical signs, eventually adding some mathematical signs (relationship with the educational goals).

3. SPECIFY: - THE KIND OF DATA YOU ANALYSED; - THE SPECIFIC ELEMENTS OF OBSERVATION.

[14] "Si equivalgono"
[15] "Sono equivalenti"

Kind of data:

Qualitative: audio and video- recordings, students' worksheets and written reports, field notes by teachers and researchers.

Specific elements of observation:

Students' and teacher's words, written signs, gestures.

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO YOUR SRQ.

   IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

As in the case of the Re-CRQ, we have made an integrated use of the data.

In particular, we have looked for relationships between students' and teacher's signs in respect to

- the DDA's characteristics on the one hand,
- the mathematical meanings underlying our educational goals on the other hand.

In the frame of the Theory of semiotic mediation, we have tried to describe the unfolding of the *semiotic potential* of Aplusix with respect to the notions of equivalence between expressions and in the acquisition of a structure sense for numerical expressions. In doing so, we have tried to describe those *semiotic chains, which* origin with the use of the artefact, to solve the task, and develop through signs that are more strictly related to such use (as the *artefact-signs*), to signs more recognizable in the domain of mathematics (what we call *mathematical signs*).

Another important element of analysis has been the action of the teacher in coordinating and leading the classroom discussions. With this respect, we have analysed how Aplusix can be exploited by the teacher as a *tool of semiotic mediation* for the envisaged educational goals, keeping in mind the background goal of introducing algebraic computation.

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

The concerns which guides our analysis are the same of the CRQ.

6. IS YOUR SRQ MEANT TO CONTRIBUTE TO PROVIDE AN ANSWER TO YOUR RE-CRQ? IF YES, HOW?

Since the SRQs are more general than our re-CRQ, which focuses in particular on the tree representation, the answer to the re-CRQ is in a sense included in the answers to the SRQs.

A.5.13 Analysis of Unisi TE with Casyopée

Validation of DDAs and PPs

(1) WERE THE EDUCATIONAL GOAL(S), SPECIFIED IN YOUR TE PORTRAITS, ACHIEVED? HOW CAN YOU ATTEST THAT?

The educational goals as stated in the PP (July '07) and cited in the TE Portrait:

The main goals envisaged when designing this Pedagogical Plan are that students construct:

7. the meaning of function as co-variation and thus consolidate (or enrich) the meanings of function they have already constructed;
8. the meaning of the processes characterizing the algebraic modelling of geometrical situation.

More specifically,
as for the notion of function, students should consolidate:

- the meaning of variables both geometrical and numerical,
- the meaning of domain of a variable,
- the meaning of function as co-variation over time of variables (even of different kind: numerical or geometrical),
- competencies related to the passage between different representations of function (at least, algebraic and graphical ones)

as for the modelling process, students should learn to:

- recognize geometrical variables,
- associate numbers (numerical variables) to geometrical variables,
- associate geometrical variables to numbers (numerical variables),
- pass from immeasurable geometrical objects (e.g. points) to measurable geometrical objects,
- parameterise (cope with the possible too high number of geometrical variables),
- express the relation between numerical variables through formulas.

The educational goals were reformulated in a way more consistent with the TF adopted and more congruent to the designed PP. The new formulation is the following (changes highlighted):

The main goals envisaged when designing this Pedagogical Plan are to foster the evolution of students' personal meanings towards:

1. the mathematical meaning of function as co-variation and thus consolidate (or enrich) the meanings of function they have already appropriated;
2. the mathematical meanings related to the processes characterizing the algebraic modelling of geometrical situation.

More specifically,
as for the notion of function, students should consolidate or enrich:

- the meaning of variables both geometrical and numerical,
- the meaning of domain of a variable,

- the meaning of function as co-variation over time (even when different kinds of variables are involved),
- competencies related to the passage between different representations of function (at least, algebraic and graphical ones)

as for the modelling process, students should learn to:

- recognize geometrical variables,
- associate numbers (numerical variables) to geometrical variables,
- associate geometrical variables to numbers (numerical variables),
- pass from not-measurable geometrical objects (e.g. points) to measurable geometrical objects,
- parameterise (optimize the number of variables),
- express the relation between numerical variables through formulas.

Moreover some remarks have to be added:

Remark 1: according to the designed pedagogical plan students were supposed to have received some formal teaching on functions, variables… thus as for the notions of "function", "variable" and related notions, the designed PP was expected to lead students to enrich the meanings they already appropriated.

Remark 2: the teacher was supposed to have some expertise in managing the class activity and in particular orchestrating collective discussions as framed within the theory of semiotic mediation.

Remark 3: we listed above many different specific educational goals in which the main educational goals are articulated. Though all those aspects could be singularly pursued through the planned PP, it is not reasonable to think to be able of pursuing all of them together. Actually, the choice of the specific educational goals to focus on, rests on the teacher. That option certainly depends also on how the activities progress.

As specified in our TE-Portrait,

> The achievement of the educational goals envisaged as well as the consistency between them and the hypotheses underpinning the Pedagogical Plan could be attested through the analysis of the semiotic processes which take place in the class. In fact we expect to be able to provide evidence of students' production of signs (namely of artefact-signs) and to trace their hypothesized evolution (the semiotic chain) towards the mathematical meanings described above. Traces of such production and evolution should be found in students' productions: solutions to the given tasks, individual reports on the activity, class discussions.

More specifically, students can be said to have achieved the envisaged educational goals if:

d. **they use specific terms (function; independent, dependent, geometrical, numerical… variable; graph; measure; domain; variation; co-variation; ecc.) in "appropriate ways" (i.e. consistently with their (possible) mathematical meanings, the DDA functionalities and the specific activities at stake);**
e. **they relate mathematical meanings and processes to the software functionalities;**
f. **they express the main phases characterizing algebraic modelling of geometrical problems.**

Evidence of students' achievement emerges from the analysis of students' reports, their written solutions to the tasks with the DDA, and the transcripts of the class discussions.

On the one hand, that analysis allows to identify expressions (constructed by students) in which specific terms (see *§1a*) are used to report on the tasks accomplished through the DDA. That witnesses that already formed personal meanings are related to or re-elaborate in the light of the actual use of the DDA (including the specific kind of tasks accomplished through it), thus testifying a progressive enrichment of students' personal meanings towards the formation of the desired mathematical meanings.

On the other hand, one can identify the use of *artefact-signs*, that is signs referring to the context of the use of the artefact, very often referring to one of its parts and/or to the action accomplished with it. These signs sprout from the activity with the artefact, their meanings are personal and commonly implicit, strictly related to the experience of the subject. But at the same time, those signs have potentialities to evolve towards mathematical signs.

Two "movements" can be attested: the use of already known mathematical terms to describe the activities with the DDA, and the use of artefact-signs in a way consistent with their mathematical potentialities. That confirms the development of a texture of meanings and signs which bridges together the artefact-world and the mathematics-world.

Summing up, we can claim that the envisaged educational goals are at least partly achieved.

As for the notion of function, variables and so on, students use specific terms and relate mathematical meanings to the DDA functionalities in appropriate ways. Though different stages are evident.

As for modelling, the idea of modelling is still related to the actual solution of specific kind of problems. Modelling in itself has not explicitly formulated yet.

(2) ON THE BASIS OF YOUR EXPERIMENTATION, CAN YOU CONFIRM THE SOUNDNESS OF THE HYPOTHESES SPECIFIED IN YOUR TE PORTRAIT, AND THE RELATIONHIP WITH THE ACHIEVEMENT OF THE EDUCATIONAL GOALS?

EXPLAIN BY MAKING REFERENCE, IF POSSIBLE, TO THE CRITERIA SPECIFIED A-PRIORI IN YOUR TE PORTRAIT.

From the TE Portrait:

The Pedagogical Plan is designed consistently with the **Theory of Semiotic Mediation**, accordingly the teaching sequence will be structured as an iteration of **didactical cycles**, constituted by the following semiotic activities: Activities with the artefacts, Individual production of signs, Collective production of signs. By **semiotic activitie**s we mean the production and elaboration of signs, related to the previous activities with artefacts.

The definition of the stated educational goals is supported by the hypothesis that Casyopee, namely the Geometrical Calculation sub-environment, can be used by the teacher as a **tool of semiotic mediation** exactly for (1) the meaning of function as co-variation and (2)the meaning of the processes characterizing the algebraic modelling of geometrical situation, as articulated above.

Such hypothesis is assumed in consequence of the analysis of the **semiotic potential** of the artefact which encompasses the analysis of the signs which the individual can produce when accomplishing specific tasks using the artefact (**artefact-signs**) and of the possible evolution (**semiotic chain**) of such signs towards mathematical signs expressing the relationship between artefact and knowledge. Hence the analysis of semiotic potential of Casyopée involves the analysis of both the personal and mathematical meanings related to the artefact, as well as of the possible tasks which can be accomplished with it. The semiotic potential of Casyopée is articulated in the semiotic potential of the specific features exploited and the meanings which could arise is articulated according to the **schemas of use** related to the specific tasks assigned.

The achievement of the educational goals envisaged as well as the consistency between them and the hypotheses underpinning the Pedagogical Plan could be attested through the analysis of the semiotic processes which take place in the class. In fact we expect to be able to provide evidence of students' production of signs (namely of artefact-signs) and to trace their hypothesized evolution (the semiotic chain) towards the mathematical meanings described above. Traces of such production and evolution should be found in students' productions: solutions to the given tasks, individual reports on the activity, class discussions.

The possible confirmation of the hypotheses inspiring the design of the PP and linking the use of the DDA with students' achievement is questioned through the Re-CRQ.

In synthesis, answering the question whether an artefact functioned as a tool of semiotic mediation for some mathematical meaning requires to investigate different but certainly related aspects:

a) the possible unfolding of the hypothesized *semiotic potential* of the artefact in relation to the designed tasks, and in relation to the target mathematical meanings;

With that respect, the written productions of the students involved in our TE show the generation of artefact-signs susceptible of (i) contributing to form a texture of meanings enriching the meanings students already appropriated and (ii) evolving towards the desired mathematical signs. Thus confirming the of the hypothesized *semiotic potential.*

b) and the possible evolution of students' personal signs towards the desired mathematical signs and the possible development of a texture of different meanings related to the target mathematical meanings which (that texture) contributes to enrich already formed personal meanings.

The evolution of student' personal signs and meaning can be attested through the analysis of students' reports, their written solutions to the tasks with the DDA, and the transcripts of the class discussions.

More in details, the evidence of that evolution is given by the identification of (i) expressions (constructed by students) in which specific terms (function, variable,…) are used to report on the tasks accomplished through the DDA; (ii) the production and use of *artefact-signs;* (iii) the production of *semiotic chains*, in which connections are established between artefact signs and mathematical signs.

As argued in the previous section, we can claim that the envisaged educational goals are at least partly achieved.

c) the possible exploitation by the teacher of the unfolded *semiotic potential* for fostering the evolution of students' signs towards the desired mathematical signs;

With that respect it is important to investigate whether and how the teacher fuel the class discussion and contributes (directly or not) to the generation of *semiotic chains* establishing connections between artefact-signs and mathematical signs. We can find examples attesting how the teacher's actions fuel the discussion, thus fostering students' construction of a semiotic chain in which there appears the development of a texture of different meanings related to the notions of variable and co-variation; but also episodes in which the teacher does not succeed to exploit the potentialities emerged from the students' interventions.

More details and examples are given and discussed in the answer to the Re-CRQ.

Common Research Question

1. REPORT YOUR RE-FORMULATION OF THE COMMON RESEARCH QUESTION (RE-CRQ)

Does the sub-environment Geometric Calculation of Casyopee function as a tool of semiotic mediation for the mathematical meaning of function as co-variation? Where the meaning of function as co-variation can be articulated in:

- the meaning of both geometrical and numerical variables,
- the meaning of domain of a variable,
- the meaning of function as co-variation over time of variables (even of different kind: numerical or geometrical).

Does the sub-environment Geometric Calculation of Casyopee function as a tool of semiotic mediation for the mathematical meanings related to the processes characterizing the algebraic modelling of geometrical situation? Where those processes encompass:

- identify geometrical variables,
- pass from immeasurable geometrical objects (e.g. points) to measurable geometrical objects,
- associate numerical variables to geometrical variables (that is define (mixed) functions from a domain of geometrical variables to a numerical set),
- associate geometrical variables to numerical variables (that is define (mixed) functions from a numerical domain to a set geometrical variables),
- parameterise (cope with the possible too high number of geometrical variables),
- express the relation between numerical variables through formulas.

2. ANSWER YOUR RE-CRQ.

WITHOUT RENOUNCING TO YOUR OWN THEORETICAL FRAMEWORK(S) AND LANGUAGE, TRY TO ARTICULATE YOUR ANSWER BY MAKING REFERENCE TO THE THREE POLES OF THE NOTION OF DIDACTICAL FUNCTIONALITY AND TO THE SHARED MINIMAL THEORETICAL FRAMEWORK.

In general, answering the question whether an artefact functioned as a tool of semiotic mediation for some mathematical meaning requires to investigate different but certainly related aspects:

the possible unfolding of the hypothesized *semiotic potential* of the artefact in relation to the designed tasks, and in relation to the target mathematical meanings;

the possible exploitation by the teacher of the unfolded *semiotic potential* for fostering the evolution of students' signs and meanings towards the desired mathematical signs and meanings;

the possible evolution of students' personal signs towards the desired mathematical signs and the possible development of a texture of different meanings related to the target mathematical meanings which (that texture) contributes to enrich already formed personal meanings.

In order to investigate the unfolding of the semiotic potential and the possible evolution of students' personal signs and meanings, we focus our attention on the signs produced and used by students in the different activities of the didactical cycles compounding the PP: tasks to be accomplished through the use of the DDA, production of written reports and class discussions.

In order to investigate whether and how the teacher exploited the unfolding of the *semiotic potential* for fostering the evolution of students' signs and meanings, we will focus on the semiotic actions which the teacher performs to orchestrate the class-discussions

a) unfolding of the semiotic potential and evolution of personal signs and meanings

The *semiotic potential of an artefact*[16] (and then its possible *unfolding*[17]) clearly depends (a) on the characteristics of the artefact (the DDA in our case), among which crucial importance has to be given to the representations provided; as well as (b) on the tasks which the individual is asked to accomplish through the artefact (and so it relates to the mode of use of the DDA).

The unfolding of the semiotic potential reveals in the production and use (by students) of *artefact-signs*[18], in ways which are pertinent to the tasks accomplished with the DDA, consistent with the DDA functionalities and consistent with mathematical potentialities of the artefact-signs themselves.

The evolution of students' personal signs and meaning towards the desired mathematical meanings (articulated in our Re-CRQ), is precisely the educational goal of the designed PP.

As discussed, in the "Synthesis of the TE", we can claim that the envisaged educational goals are partly achieved. More in details, as for the notion of function, variables and so on, students use specific terms and relate mathematical meanings to the DDA functionalities in appropriate ways. Though different stages are evident. As for modelling, the idea of modelling is still related to the actual solution of specific kind of problems. Modelling in itself has not explicitly formulated yet.

[16] The *semiotic potential of an artefact in relation to a task* encompasses the complex of personal and mathematical signs related to the artefact and its use for accomplishing the task.
[17] The *unfolding of the semiotic potential of an artefact in relation to a task* consists in the generation and use of signs related to the artefact and its use for accomplishing the task.
[18] A*rtefact-signs* are signs referring to the context of the use of the artefact, which very often refer to one of its parts and/or to the action accomplished with it. Those signs sprout from the activity with the artefact, their meanings are personal and commonly implicit, strictly related to the experience of the subject. But at the same time, those signs have potentialities to evolve towards mathematical signs.

Evidences supporting the claimed achievements of students are given from the analysis of students' reports, their written solutions to the tasks with the DDA, and the transcripts of the class discussions.

On the one hand, that analysis allows to identify expressions (constructed by students) in which specific terms (function, variable,…) are used to report on the tasks accomplished through the DDA. That witnesses that already formed personal meanings are related to or re-elaborate in the light of the actual use of the DDA (including the specific kind of tasks accomplished through it), thus testifying a progressive enrichment of students' personal meanings towards the formation of the desired mathematical meanings.

On the other hand, one can identify the use of *artefact-signs*, that is signs referring to the context of the use of the artefact, very often referring to one of its parts and/or to the action accomplished with it. These signs sprout from the activity with the artefact, their meanings are personal and commonly implicit, strictly related to the experience of the subject. But at the same time, those signs have potentialities to evolve towards mathematical signs.

Two "movements" can be attested: the use of already known mathematical terms to describe the activities with the DDA, and the use of artefact-signs in a way consistent with their mathematical potentialities. That confirms the development of a texture of meanings and signs which bridges together the artefact-world and the mathematics-world.

In particular, we can identify the generation of *semiotic chains*[19] showing how connections are established between artefact-signs and mathematical signs.

<u>Unfolding of the semiotic potential and evolution of personal signs and meanings: evidences</u>

Hereafter, we will report excerpts relating to the different kinds of activity which characterized the implemented PP: students' activity with the DDA, class discussion, students' personal reports on the class activities.

1. The unfolding of the semiotic potential may be attested through the identification of the generation and use of artefact-signs in students' written solutions of the given tasks, reports on the class activities and interventions in the class discussions.

2. The following excerpt is drawn from the transcript of the class discussion held in the 5th session (1st session: familiarization with the DDA; 2nd section: optimization problem to be solved through the DDA; 3rd session: class discussion; 4th session: optimization problem to be solved through the DDA). It shows an example of how artefacts signs are produced in relation to the use of the artefact, and how they evolve during the discussion.

1.	**T:** "Which are the main points to approach this kind of problem? Which kind of problem did we deal with? […] What is an important thing you should do now? To see the general aspects and apply them for solving possible more problems, […] the software guided you proposing specific points to focus on.[…]

[19] By semiotic *semiotic chain*, we mean a chain of signification "in which the external reference is suppressed and yet held there by its place in a gradually shifting signifying chain." (Walkerdine, 1990, p.121).

…

5. **Luc**: " you have to choose a mobile point, first […]"

…

16. **T:** " […] do you see anything similar between the two problems?"
17. **Sam:** "one has always to take a free point which vary, in this case, the areas considered […]"
18. **T:** "Then we have a figure which is…"
19. **Students:** "Mobile."
20. **T:** "Mobile, dynamical. Let us pass to the second phase. Andrea, which is the next phase? […]"
21. **And:** "[…] we need to study that figure and observe what the shift of the variable causes…"
22. **T:** "ok, then? Anybody did that, isn't it?"
23. **Sil:** "[…] by shifting the mobile point one observed as [the sum of the areas] varied"

We can notice:

I. The collective construction of a *semiotic chain*, in which a connection is established between artefact signs ("mobile point") and mathematical signs ("variable"). The elements of this semiotic chain are: "movable point", "free point", "variable", and "movable point". It is worth noticing the two directions: from the artefact sign ("mobile point") to the mathematical sign ("variable") and viceversa. That semiotic chain shows: (a) students' recognition that geometrical objects can be considered (can be treated, can act as) as variables (b) the enrichment of students' meanings of variable to include meanings related to "movement".

II. Elements of a semiotic chain in which the meaning of function as a relation of co-variation of two variables emerges. The elements of this semiotic chain are: "a free point which vary […] the areas" -> "the shift of the variable causes" -> "by shifting the movable point, one observed as [the sum of the areas] varied" … more elements can be found in the continuation of the discussion.

3. Hereafter there is an excerpt from Valeria's 5th report (homework, after the 2nd class discussion, 5th session)

What do you mean by the terms "function", "independent variable" and "dependent variable"?

[…] The independent variable is the one which is modified first, as a consequence of that the other one [the dependent variable] is modified. […]

Which elements of the software can be put in relationship with those terms? Why?

The independent variable corresponds to the mobile point, because it is the element which can be arbitrarily modified, whereas all the figures […] are dependent variables, because their area and perimeter are modified according to how the mobile point is shifted.

The above excerpt can be analysed at least at two different levels.

On the one hand, we can consider Valeria's answers separately. They are both "consistent" in themselves (though not complete): the former is pertinent to the mathematical meanings at stake, and the latter is pertinent to the DDA functionalities and the tasks accomplished through it. Moreover mathematical signs (“independent variable”, “dependent variable”) and artefact signs (“mobile point”, “figure”, “area”, “perimeter”, “shift”, “modify”) are consistently used.

On the other hand, if we compare the two answers we can notice an impressive semiotic correspondence between them. Such correspondence reveals the establishment of a consistent relationship between the signs “independent variable” (mathematical sign) and “mobile point” (artefact sign), and “dependent variable” (mathematical sign) and “figure”, “area” and “perimeter” (artefact signs), and therefore between the associated meanings.

Finally, from both the answers the meaning of function as co-variation emerges too (“as a consequence”, “according to”).

It is not possible to carry out a so fine-grained semiotic analysis, for every students’ written productions. And certainly, there are differences between the students’ achievements.

But, globally, we can claim that the envisaged educational goals are at least partly achieved.

b) teacher’ exploitation of the unfolded semiotic potential

By teacher’s exploitation of the unfolding of the semiotic potential we mean the complex of strategies which the teacher enacts to foster the evolution of students’ personal signs and meanings towards the desired mathematical signs and meanings (the designed educational goals). In the previous section, we focused on how the DDA was used by the students for accomplishing specific tasks. Now we are introducing a new “dimension” of use of a DDA: the teacher’s use of the DDA.

In fact, in order to foster the evolution of meanings, the teacher may recall the context of use of the DDA, ask students to establish explicit connections between the emerging mathematical signs and the DDA features (e.g. commands, representation), and so on. When the teacher succeed in exploiting those potentialities, we say that the artefact is used as a *tool of semiotic mediation.*

The analysis of the teacher’s exploitation of the semiotic potential of an artefact requires the study of the strategies enacted by the teacher to facilitate the evolution of students’ personal signs. With that respect it is important to investigate whether and how the teacher fuels the class discussion and contributes (directly or not) to the generation of *semiotic chains* in which connections between artefact-signs and mathematical signs are established.

Hereafter we discuss two examples. The former (the same of a previous section) shows how the teacher’s actions fuel the discussion, thus fostering students’ construction of a semiotic chain in which there is an apparent development of a texture of different meanings related to the notions of variable and co-variation. The latter shows an episode in which the teacher does not succeed to exploit the potentialities emerged from the students’ interventions.

Teachers’ exploitation of the unfolded semiotic potential: evidence.

The following excerpt is drawn from the transcript of the class discussion held in the 5th session. It has been previously analysed form the point of view of the signs produced and used by students. Here we focus on how the teacher's actions foster the production of artefacts signs in relation to the use of the artefact, and create the conditions for their evolution during the discussion.

1.	**T:** "Which are the main points to approach this kind of problem? Which kind of problem did we deal with? […] What is an important thing you should do now? To see the general aspects and apply them for solving possible more problems, […] the software guided you proposing specific points to focus on.[…]
…	
5.	**Luc**: " you have to choose a mobile point, first […]"
…	
8.	**T:** " […] do you see anything similar between the two problems?"
9.	**Sam:** "one has always to take a free point which vary, in this case, the areas considered […]"
10.	**T:** "Then we have a figure which is…"
11.	**Students:** "Mobile."
12.	**T:** "Mobile, dynamical. Let us pass to the second phase. Andrea, which is the next phase? […]"
13.	**And:** "[…] we need to study that figure and observe what the shift of the variable causes…"
14.	**T:** "ok, then? Anybody did that, isn't it?"
15.	**Sil:** "[…] by shifting the mobile point one observed as [the sum of the areas] varied"

First of all, the teacher asks students to report on their solutions to the problem dealt with in the previous sections. She explicitly orients the discussion towards the specification of the main phases of the solution of the problem (item 1), asking students to look for similarities between the two problems addressed so far and between strategies enacted to solve them (items 1 and 8). While asking students to do that, the teacher suggests to refer to (or to remind) the use of the DDA (item 1).

The suggestion to explicitly refer to the use of the DDA, facilitates the production and use of artefact-signs and the unfolding of the semiotic potential. At the same time the request to generalize (though a little vague) fosters a de-contextualization from the specific problems faced and strategies enacted and provides the possibilities for the evolution of personal signs and meanings to initiate.

Then we can notice how the teacher's interventions fuels the discussion: the teacher introduces the term "figure" (item 10) which on the one hand has the effect to maintain the focus still on geometrical objects, contrasting a student's tendency to prematurely (from the teacher's viewpoint) shift the attention on numbers; and on the other one offers the possibility to re-introduce the consideration of "dynamical aspects" (also reprised by the teacher in her subsequent intervention, item 12) which fuels the construction of a semiotic chain on variation and co-variation.

That is an example of what we mean by saying that the teacher uses the artifact as a tool of semiotic mediation.

The following excerpt, on the contrary, shows an episode in which the teacher does not succeed to exploit the potentialities of the students' interventions. Chi countered "variable" with “variable point” so offering the possibility to dwell on the relationship between not measurable geometrical variables and measurable geometrical variables which was considered a key aspect of algebraic modeling. The teacher does not foster any discussion on that, she was probably aiming at orienting the discussion along a different direction.

203.**Lor**: a mobile point on the side […]

204.**Chi**: then, when we had to calculate the area… well meanwhile we put CD as *x*, we set a variable *x*

…

208.**Chi**: we put CD as variable, and not by chance CD, in fact we used a fixed point, C, and a variable point on the segment, D

209.**T:** well, the underpinning idea is to link numbers, and, […] having observed a link between the position of the point D and […] the area of the rectangle […] a link is established between a geometrical world and an algebraic world

That witnesses the difficulty of mobilizing strategies to foster the evolution of students’ signs. In fact the evolution of students’ signs depends on extemporary stimuli asking for a number of decision on the spot.

3. SPECIFY:

- THE KIND OF DATA YOU ANALYSED;
- THE SPECIFIC ELEMENTS OF OBSERVATION.

KIND OF DATA ANALYSED: mainly students’ written productions (detailed below) and transcripts of class discussion.

Details about students’written productions: students worked with the DDA in small groups (2 students as far as possible, 3 students occasionally) and were asked to produce common written documents related to the tasks accomplished through the use of the DDA: e.g. solutions to given mathematical problems, and comments on the use of the DDA for solving those problems.

In addition, at the end of each session students were asked to individually write at home a report concerning the work with the DDA, and based on a small set of questions.

In order to better document students’ actual work with the DDA we also gathered other kinds of data: DDA log files produced by students, and video-records of some students’ desktops. Those data were used as a complement of the analysis of students’ written productions.

ELEMENTS OF OBSERVATION:

Our analysis focused on the signs (relevant to the designed educational goals) generated and used by students in the different sessions of the PP, as well as on the semiotic strategies enacted by the teacher to foster the evolution of students’ personal signs.

More in details we tried to identify possible artefact-signs, mathematical signs, *hybrid signs or sentences*[20], *pivot signs*[21], as well as possible semiotic chains connecting those signs. (We have not dwelt yet on the hybrid (or pivot) character of the signs generated in the class, we will discuss that in the future)

4. DESCRIBE HOW THE ELEMENTS OF OBSERVATION WERE USED TO SUPPORT YOUR ANSWER TO THE RE-CRQ. IF POSSIBLE, MAKE EXPLICIT WHICH ELEMENTS OF YOUR THEORETICAL FRAMEWORK(S) WERE USED IN THE ANALYSIS PROCESS AND HOW.

I. The unfolding of the semiotic potential reveals in the production and use (by students) of *artefact-signs*, in ways which are pertinent to the tasks accomplished with the DDA, consistent with the DDA functionalities and consistent with mathematical potentialities of the artefact-signs themselves.

Evidences supporting the claimed achievements of students are given from the analysis of students' reports, their written solutions to the tasks with the DDA, and the transcripts of the class discussions.

On the one hand, that analysis allows to identify expressions (constructed by students) in which specific terms (function, variable,…) are used to report on the tasks accomplished through the DDA. That witnesses that already formed personal meanings are related to or re-elaborate in the light of the actual use of the DDA (including the specific kind of tasks accomplished through it), thus testifying a progressive enrichment of students' personal meanings towards the formation of the desired mathematical meanings.

On the other hand, one can identify the use of *artefact-signs*, that is signs referring to the context of the use of the artefact, very often referring to one of its parts and/or to the action accomplished with it. These signs sprout from the activity with the artefact, their meanings are personal and commonly implicit, strictly related to the experience of the subject. But at the same time, those signs have potentialities to evolve towards mathematical signs.

II. The analysis of the teacher's exploitation of the semiotic potential of an artefact requires the study of the strategies enacted by the teacher to facilitate the evolution of students' personal signs. With that respect it is important to investigate whether and how the teacher fuel the class discussion and contributes (directly or not) to the generation of *semiotic chains* establishing connections between artefact-signs and mathematical signs.

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[20] A sentence or even a single word which combines elements coming from the artefact and from the mathematics

[21] A *pivot sign* has at the same time a reference in the mathematical context and a reference in the artefact context

5. MAKE EXPLICIT WHICH CONCERNS GUIDED YOUR ANALYSIS PROCESS AND HOW.

a) Characteristics of the DDA

a.1 concerns about the ways mathematical objects and their interaction are represented,

a.3 concerns about the ways representations can be acted on.

They both induce us to focus on the signs produced by students in relation to the activities with the artefact. They guide us in identifying artefact signs.

b) Educational goals

b.1 epistemological concerns.

b.2 semiotic concerns.

They both guide us in the analysis of the evolution of students' signs and meanings towards the desired mathematical signs and meaning. Epistemological concerns guide us more in analysing students' achievements from the point of view of the mathematical knowledge of reference. Semiotic concerns orient especially our analysis of the evolution (modification) of students' personal signs and meanings.

c) Modalities of use

c.1 concerns about the tasks and their temporal organization,

induce us at focusing on the signs produced in the different activities and how such signs relate to the specific tasks.

c.2 concerns about the functions to be given to the DDA and their possible changes,

induce us to pay attention: (a) on the link between the different signs students produced and the students' use of the DDA for accomplishing the tasks; (b) on the link between the evolution of students' personal signs and the possible different function which the DDA assumes when it is used as a tool of semiotic mediation by the teacher.

c.3 concerns about semiotic issues,

are the key concerns in our analysis, they lead us to analyse signs and what is connected to their "transformation".

c.4 concerns about the relationship between knowledge referred to the DDA functioning and knowledge referred to the educational goals,

guide us in analysing the unfolding of the semiotic potential.

c.5 concerns about social organization and interactions,

lead us to differentiate our analysis according to the kind of action performed by the different actors with respect to signs, their production and their evolution.