## Appendix 2: Cross-case Analyses

## I. Aplusix cross-case analysis

## I.1. Identification

The teams involved are:
UJF team (France): familiar DDA
ITD team (Italy): alien DDA
UNISI team (Italy): alien DDA
DDA considered:

## I.2. Contextual elements

School level:

- 28 students Grade 10 (15-16 years) - UJF
- 2 classes ( 26 students and 29 students) Grade 9 (14-15 years) - UNISI
- 14 Students of Grade 7 (11-12 years) - ITD

Physical context:

- Classroom equipped with computers, overhead projector.
- Students work in pairs.

For UNISI and ITD: Students were sometimes involved in collective discussions
Length of the scenario:

- UJF: 5,5 hours
- UNISI: 18 hours
- ITD: 10 hours

The teacher involved in the ITD experiment took part to the design of the experiment collaborating with the ITD team. The experiment was designed to be inserted in the curriculum of the class. It was well accepted by the institutional context of the school.

The teacher involved in the UJF experiment didn't take part to the design of the scenario but he had a possibility to adapt it to the constraints of his class.

Two teachers were involved in the UNISI experiment. One of them took part to the design of the experiment and she was supposed to be familiar with the use of technology in class because she has been part of the group for years. The other teacher was a young teacher who hadn't collaborated
with the team before. She didn't participate to the design of the experiment, but she only implemented it in classroom.

## I.3. Theoretical frames

The three teaching experiments share a common semiotic concern and the theoretical construct of Semiotic register of representation (Duval, 1995, 2006) can be considered a common component of the three theoretical frameworks. In this perspective, three semiotic registers are used in each PP: standard representation (SR), tree representation (TR) and natural language representation (NL). A common hypothesis concerns the role of conversion tasks to make the meaning of structure of an algebraic expression emerge. Further elaborations of this semiotic approach are developed in the three different theoretical frameworks assumed by each team and presented in the following.

UJF used the Anthropological Theory of Didactics (Chevallard, 1999). He identifies 6 moments in studying a given type of task (what he calls didactic organisation): (1) first encounter with the type of task, (2) exploring the type of task and emergence of a technique, (3) constructing the technological-theoretical unit, (4) institutionalisation, (5) working out the technique, and (6) assessment. The familiar PP is organized in accordance with such a didactic organization

UNISI used the Semiotic Mediation Theory (Bartolini Bussi \& Mariotti, 2008) A basic assumption of the theory is that mathematical meanings are rooted in the action with the artefact, and developed through social interaction in classroom. Identifying the semiotic potential of an artefact is the starting point for developing a teaching/learning sequence which involves the use of such artefact. This means to identify the potential that an artefact has with respect to some mathematical meanings, in relation to the tasks in which it is used. Referring to the alien PP, feedback signs provided by Aplusix have been considered as having a semiotic potential respect to the meaning of equivalence of expressions.

ITD used the Activity Theory (Cole \& Engeström, 1991). According to this theory, learning can emerge overcoming contradictions that can appear during educational activities. The tasks of the ITD's PP are designed to be source of contradiction through the comparison of solutions performed in paper and pencil and the solutions performed in Aplusix. Feedback provided by Aplusix is crucial to make emerge this contradiction.

## I.4. Comparison of didactical functionalities

## The educational goal

Among the educational goals of the three experiments a common element is constituted by the achievement of what is called structure sense (Hoch and Dreyfus, 2006).
"A student is said to display structure sense for high school algebra if s/he can:

- Recognise a familiar structure in its simplest form.
- Deal with a compound term as a single entity, and through an appropriate substitution recognise a familiar structure in a more complex form.

Choose appropriate manipulations to make best use of a structure."
(Hoch and Dreyfus, 2006)

The achievement of this common educational goal is related to the basic meaning on which the algebraic calculation is rooted: the equivalence between algebraic expressions. In fact, the equivalence between expressions 'passes through' different structures.

## The characteristics of Aplusix

Aplusix is a computer algebra system which allows students to perform both arithmetical and algebraic calculations (Nicaud \& al., 2004). Adopting the terminology introduced by Duval (1995, 2006), Aplusix offers three different registers of representation of algebraic expressions on which to act: natural language, standard representation, which is the usual representation of expressions in paper and pencil, and tree representation ${ }^{1}$.

Three modes of tree editing have been implemented into Aplusix: free tree mode, controlled tree mode, and mixed tree mode. While in free tree mode, expressions can be edited freely as trees and no constraints on the tree are provided during the editing process, in the controlled tree mode the system provides constraints to the editing process, preventing the user from constructing syntactically incorrect trees. For instance, because the arity of operators must be correct, it is not possible to build a tree made of three branches with the minus sign as operator. Both in free and controlled modes, the system only accepts trees in which internal nodes are operators and leaves are numbers or variables. The mixed tree mode constitutes a hybrid representation that combines both the standard and the tree representations. In fact, a standard representation can be expanded as a tree by clicking on the " + " button that appears when the mouse cursor is on the left side of a node (Fig. 1a, b); vice versa, a tree, or a sub-tree, can be collapsed into a standard representation by clicking on the "-" button (Fig. 1c). Thus, mixed representation presents a scaffolding feature, besides the syntactical constraints that characterise the controlled mode.


Figure 1. Mixed tree representation: (a) the expression ( $x-2$ )( $2 x+1$ ) in standard representation can be developed into a tree by clicking on the " + " button; (b) the expression after a first level development;
(c) "-" button allows collapsing the tree into standard representation, as in Fig. 1a.

[^0]All the tasks in Aplusix, can be performed both in training and in test mode. The training mode is characterized by a feedback provided by the DDA. The feedback is based on the on the equivalence between the algebraic expressions produced in two consequent steps. Feedback ${ }^{2}$ is expressed by means of three different signs (Fig. 2).


The black lines point out that the expressions are equivalent, the red crossed lines that they are not equivalent, and the blue crossed lines indicate that the edited expression is not well formed (e.g., in Fig. 2, the last term has not been entered yet). In the test activity, no feedback is provided: at each stage a single black line links two consequent steps. Finally, the observation modality allows the student, the teacher or the researcher to replay the whole sequence of steps performed by the user in order to solve the task. The comparison between the three Teaching Experiments presented in this paper focuses on the DDA feature consisting in the feedback provided in the training activity.

## The modalities of employment

Though starting from common educational goals, the great variety of possible choices offered by Aplusix features - two modes of feedback control, three modalities of tree editing, ...- provided a variety of modes of use. Actually, in the design of the PPs, the three teams showed a great variety of choices that can be explained according to the three different theoretical perspectives they refer to.

In the following, the familiar PP and two alien PPs are compared highlighting the different modalities of use of the feedback as they are developed in the three experiments. The familiar PP (UJF team) refers to the Anthropological Theory of Didactics (Chevallard, 1999); one of the alien PP (UNISI team) has the Semiotic Mediation (Bartolini Bussi \& Mariotti, 2008) as a leading theory of reference and the other PP (ITD team) has been designed following the Activity theory framework.

## I.5. Results of the cross-case analysis together with illustrative examples

## The UJF pedagogical plan and its results

The familiar PP has been designed to be experimented by teachers who use Aplusix at a regular basis in their classrooms. Therefore the students are familiar with the system, apart from the tree representation of expressions that is novel for them. Starting from the assumption that tree representation of algebraic expressions highlights their structure, the main educational goal of the

[^1]PP is to use this representation to study the structural aspect of algebraic expressions and thus help students to distinguish between procedural and structural aspects of expressions. The core of the PP is organized in three main phases:

- familiarization phase, which is guided by the teacher who introduces the tree representation through the mixed representation. Starting from a simple expression in standard representation, she/he expands the nodes so as to obtain a tree (a controlled tree);
- interaction between natural language and tree representations;
- interaction between usual and tree representations.

In the phases 2 and 3 , students will first encounter controlled tree mode. Thus, they cannot make syntactical errors in building trees, since the constraints of the software will prevent them from doing so. Later, students are proposed similar tasks in the free tree mode, i.e., there are no constraints during the editing process, but the students benefit from the feedback provided by the system allowing them to check the correctness of their solutions.

The choice of making students deal first with controlled trees in most of the sessions of the PP and encounter the free trees afterwards can be explained according to the adopted theoretical framework, which is Chevallard's Anthropological theory of didactics (Chevallard, 1999).

In fact, Chevallard identifies 6 moments in what he calls didactic organisation, accordingly, six types of task in studying a given type of task (what he calls didactic organisation): (1) first encounter with the type of task, (2) exploring the type of task and emergence of a technique, (3) constructing the technological-theoretical unit, (4) institutionalisation, (5) working out the technique, and (6) assessment. The familiar PP is organized in accordance with such a didactic organization. During the familiarization phase, for the first time (moment 1), the students will encounter the tree representation, and more specifically the task of conversion of an algebraic expression given in a standard representation into a tree. The teacher will be orchestrating the class discussion aiming at making the way the tree is developed emerge (moment 2). Several examples of algebraic expressions involving different operators are worked out by the students and collectively discussed, under the teacher's orchestration (moment 3). The familiarization process will end by an institutionalisation phase where the appropriate vocabulary is introduced and the technique of building a tree is formulated (moment 4). In the next session, the students are given tasks consisting in building trees representing expressions given either in natural language or in standard representation. They are supposed to work first in the controlled mode. This moment corresponds to the moment 5 in Chevallard's didactic organization. Since students are requested to solve a new type of task for the first time on their own, supporting them with a kind of scaffolding coming from the functioning of controlled mode (that is preventing them from committing syntax errors) seems to be promising. Afterwards, similar tasks will be given to the students, but this time they will work in free tree mode, hence they will not benefit from scaffolding anymore. This choice is motivated by our wish to assess students' mastery both of the tree representation and of conversion tasks (moment 6). The following example shows how the controlled mode may assist the student in building a tree representation of an algebraic expression given by a description in natural language. In solving the task 'build a tree corresponding to the expression "y squared"', a student proceeded as shown in figure 3 .

The student entered the symbol "^"" which is used in Aplusix for the power operator. The system created two branches with question marks as leaves (Fig. 3a). In the first attempt, the student seemed to be proceeding from left to right (in French, the expression is read "carré de y", i.e., "square of $y$ "). Thus, he entered 2 to the left hand branch and y to the right hand one (Fig. 3b). Observing the feedback provided by Aplusix, he realised that the tree was not correct. The second attempt (Fig. 3c) can be seen either as the student's interpretation of $y^{2}$ as $y \times y$, or as the intermediate step towards the correct tree obtained in the third attempt (Fig. 3d).


Figure 3. Three attempts to build a tree representation of the expression "y squared".
This example shows that such scaffolding can help students master more easily and quickly the new register and conversion tasks, so that they will be ready to approach new kinds of tasks, namely treatment and formation tasks (Duval, 1995).

## The UNISI pedagogical plan and its results

The alien PP is underpinned by a theory of Vygotskian perspective called Theory of Semiotic Mediation (Bartolini Bussi \& Mariotti, 2008), which aims at modelling the teaching-learning processes based on the use of artefacts.

A basic assumption of the theory is that mathematical meanings are rooted in the action with the artefact, and developed through social interaction in classroom. Identifying the semiotic potential of an artefact is the starting point for developing a teaching/learning sequence which involves the use of such artefact. This means to identify the potential that an artefact has with respect to some mathematical meanings, in relation to the tasks in which it is used. Referring to the alien PP, feedback signs provided by Aplusix have been considered as having a semiotic potential with respect to the meaning of equivalence of expressions. However, the semiotic mediation function of an artefact is not automatically activated by the use of an artefact: it's up to the teacher, who has the awareness of the semiotic potential of the artefact in terms of mathematical meanings, to foster the process of production and evolution of signs centred on the use of an artefact.

Within this theoretical perspective, in the UNISI's PP the first encounter with the software is devoted to make students conscious of the different kinds of signs provided by the DDA. According to Peirce (Peirce, 1931), a sign consists of three components: the sign or representamen (that represents), the object (that is represented), and the interpretant (that is related to the interpretation process). A basic assumption is that the object can never present itself directly to a knower: it is always mediated by a sign, of which the object is the referent. Such a frame is particularly suitable to analyse how we deal with mathematics, an abstract discipline in which the "objects" are treated
through signs. In our specific case, this reference frame appears suitable for analysing the signs provided by Aplusix as feedbacks to the students' actions, and related to the mathematical meaning of equivalence between algebraic expressions.

The feedback-signs provided by Aplusix have a twofold meaning. We can refer to them using the terms primary interpretation and developed interpretation. Let us consider the sign 'red crossed lines'. Its meaning is rooted in a social convention which can be reinterpreted in the school context; in fact, both presences of the colour red in the inscription refer to the sign of error which have the red colour and the cross in the set of its representations. Whereas the primary interpretation could refer to common sense, the developed encoding refers to a mathematical knowledge and for its nature it is not immediate or immediately shared. Since the reaction of the machine is coherent with the mathematical knowledge at all times, that makes the feedback-signs a possible instrument of semiotic mediation for the meaning of equivalence between algebraic expressions.

The link between the developed interpretation of the feedback signs is not automatic, but it indeed requires the mediation of the teacher to be developed.

This issue is addressed since the first encounter with the software, in the familiarization phase. After some brief instructions on how to open files in Aplusix and typing expressions, students are requested to accomplish a task of numerical computation. Students are also asked to interpret the different signs appearing on the screen. Specifically they are asked to take note on a sheet of paper of how such signs change during the solution of the tasks within Aplusix, and attempt a possible interpretation. Fig. 4 reports an example of students' answers to the question "Try to explain the meaning of each of the signs that appear between two lines while you are editing":

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11 = quando léesercizio è givsto compare questo seg no
\#
\# = quando éesercirio e é incompleto
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Figure 4. When the exercise is correct this sign appears.
When the exercise is not correct, is wrong.
When the exercise is not completed.
Then, during a collective discussion orchestrated by the teacher, students' interpretations of the feedback signs are shared while their mathematical meaning is expected to emerge in relation to the notion of equivalence.

The tree representation is introduced in the subsequent activity. The idea is that the new signs will be coordinated with the old ones; in particular, the standard representation system will be put into relation with the tree representation system, while the control signs system, interacting with both of them is expected to fundamentally contribute to build and consolidate the meanings related to algebraic expressions. These two activities are followed by another classroom discussion, which concludes the familiarization phase. The discussion plays an essential part in the PP: the teacher has the responsibility to guide the evolution of meanings emerging in the activity with the DDA towards meanings that are consistent with the mathematical theory. In one of the classrooms observed during the teaching experiment, many students interpreted the feedbacks of Aplusix as
shown in Fig. 4, i.e. they interpreted black lines as indicating 'correct' exercise, red crossed lines as indicating 'incorrect' exercise, and blue crossed line as indicating 'incomplete' exercise. The reference to mathematical meaning slowly emerged after the intervention of the teacher.

By analysing students' ongoing production through semiotic lens, we have been able to identify key elements that provide evidence of the role played by Aplusix components, in particular by the feedback, both in students' learning processes, and in the teaching strategy.

We report on an excerpt from a collective discussion that followed the first activity with the DDA. Students are requested to work in pairs in Aplusix to accomplish a task of numerical calculation. As already said, when the students manipulate an expression, Aplusix constantly provides a feedback related to the mathematical meaning of equivalence between expressions. The link between such kinds of feedback and its mathematical meaning is not automatic, but it is indeed a matter of interpretation. It is just to stress their semiotic nature that we have introduced the term feedbacksigns. The main goal of this first activity consists in making students interpret the three feedbacksigns; students are therefore requested to observe the feedback given by Aplusix during calculation tasks. They are also asked to take note on a sheet of paper of how the feedback-signs change during the development of the calculation, providing a meaning for each of them. In the collective discussion following the activity with the artefact, the teacher aims at exploiting the semiotic potential of the feedback-signs and intends on making the students aware of the mathematical meanings of the feedback-signs. Here after an excerpt from the discussion.

## Excerpt 1



Figure 5. The teacher's hand as picking something at two different heights.

1. Teacher: Have you all finished? There were four questions, the first asked to note down the signs appearing between a line (gesture as in Fig. 5 on the left) and the following one, What are the signs appearing between a line and the following one? gesture as in (Figure 5 on the right) .
2. Mattia: The first sign appearing when you write in the second passage, is...two vertical lines with an x over them (gestures as in Fig. 6).
3. Teacher: an $x$, two vertical lines with an x over them.
4. Davide: parallel.
5. Mattia: and then, when you complete the passage, the x disappears.
6. Teacher: the x disappears. Is what he has said right? (The teacher draws on the blackboard the three signs) [...]
7. Amalia: Well, there are three signs...well, those two vertical lines are when the passage is right and concluded [...]
8. Teacher: What does it mean "to be right"?
9. Martina: That you didn't make any mistakes in the calculations.
10. Amalia: That you have not mistaken anything and you can go to the following passage [...]
11. Teacher: And how can we do that not using the computer, understand that things are right without seeing the signs? Why are they right?
12. Ambra: Because if the calculation follows a logical thread, it is right.
13. Teacher: Because if the calculation follows a logical thread, it is right. What does it mean to follow a logical thread?
14. Martina: To do certain operations [...]
15. Teacher: Why are passages right? What does it mean to have the passages right? Where does the logical thread lead? [...]
16. Amalia: Because basically the last passage must give you the result of the first one.
17. Teacher: The last passage must give you the result of the first one: what does it mean?
18. Amalia: Yes because basically if you solve the first passage the result must be...equal to the second.
19. Teacher: Let's help her to say it well [...]
20. Valentina: Yes because finally the result is the simplification of the first, each passage has the same result.
21. Teacher: and so?
22. Amalia: Basically, if we have...I don't know... $6 / 3$ and we reduce it to the minimal terms it comes 2, doesn't it? (The teacher writes on the blackboard $6 / 3$ and 2 , side each other) So I tell that 2 is the result of the first passage [...]
23. Teacher: [...] How do we say that the result of $6 / 3$ is 2 ? In mathematics, when we speak, how can we say that the result of $6 / 3$ is 2 ?
24. Cora: That the result of 6 divided by 3 gives 2 .
25. Teacher: Yes, but...what do we say of these two (pointing to $6 / 3$ and 2 with the two hands, Fig. 7) here?


Figure 7. The teacher pointing 9 to $6 / 3$ and 2.
82. Valentina: That they are equivalent to each other
83. Teacher: That?
84. Valentina: Yes, that they are equivalent to one another, they are equivalent.
85. Teacher: And what does it mean that they are equivalent?
86. Amalia: That they are equal...

## 87. Students: That they have the same value.

From the beginning of the discussion the teacher focuses attention on the interpretation of the feedback-signs of Aplusix (\#1). As emerging from the discussion (\#1-19), and confirmed by the collected written sheets, all the students' interpret the feedback-sign as "right passage" (see \#19: "Two vertical lines [...] when the passage is right and concluded"). The personal meanings that students develop from the first activity with the artefact are consistent with the primary interpretation of the feedback. According to the classification provided by the Theory of Semiotic Mediation, the inscription " $\|$ " can be considered an artefact-sign, since its meaning is strictly related to the activity with the artefact. Under the guidance of the teacher it becomes the first element of a semiotic chain leading to the mathematical sign, referring to the notion of equivalence. Once it happened, the feedback-sign "||" has reached the level of the developed interpretation. In fact in the excerpt we can observe the following evolution for the interpretation of Aplusix feedback-sign "||":
right / no mistakes (\#19-41) becomes passages with the same result (\#67-73) becomes equivalence between passages (\#82-84)

This semiotic chain comes into existence thanks to a didactic strategy that starting from the activity with the artefact is focused on the students' semiotic processes. This strategy uses, in a synergic way, different kinds of semiotic resources: speech, gestures (an example is in \#1, Fig. 6, and the same kind of gesture-speech enactment is widespread in the whole protocol), and inscriptions on the blackboard (\#81, Fig. 7). In particular, the teacher constantly stimulates the students to make the meanings of the involved signs explicit ('what does it mean?', \#39, 62, 66, 68), to elaborate from the emerging contributions ('Let's help her to say it well', \#70; 'and so?', \#74), and to detach from the artefact ('how can we do that not using the computer', \#60) to relate to mathematics domain ('in mathematics, when we speak, how can we say that', \#79). Beyond the recurrent typical semiotic question "what does it mean", the teacher's strategy encompasses sentences and actions that have the functions of echoing and amplifying some students' contributions to the whole classroom (\#3, 6, $62,68,83$ ), and generally focusing attention towards certain elements (see for instance the deictic gesture in Fig. 7). By repeating and re-formulating students' contributions on the one hand, and making explicit reference to mathematics domain on the other hand, the teacher fosters the weaving of a texture of meanings in which the meaning of equivalence comes to be sided and overlapped to that of the right passage. This double interpretation of the feedback-signs emerging from Aplusix is the core of the semiotic potential of this specific feature of the software in solving a given task. In the following excerpt, from a discussion occurring a week later, we can see how this texture of
meaning is correctly managed by the students and referred to the artefact-signs ('black equal' and 'red equal').

## Excerpt 2

1. Mattia: Aplusix uses some symbols, for instance when we write, and make a new passage: when we finish writing the passage, if the passage is equivalent to the previous one, and therefore it is right, we have a symbol telling us that it is right, whereas if the passage that we have written is wrong with respect to the previous passage, we have another symbol.
2. Teacher: So he is saying that if we have two different expressions that are equivalent then we have in Aplusix a symbol that is?


Figure 8. Two hands mimicking the two bars of Aplusix feedback-sign.
3. Davide: the black equal
4. Teacher: the black equal, two bars (gesture as in Fig. 8). If on the contrary these two expressions are not equivalent
5. Davide: it comes the red equal

As in many other cases in the protocols (see also above) we observe how the teacher uses, in a synergic way, different semiotic resources: in this case, the utterance is accompanied by a gesture that depicts the feedback-sign provided by the software. As it has been pointed out by many researches on the role of gestures in mathematics learning (see for instance Arzarello \& al., 2009), a strict coordination of the various resources is found in the students' as well (e.g. see \# 2, Fig. 6).

In summary, in terms of didactical functionalites of the DDA, we can say that our results give evidence of the possibility of unfolding of the semiotic potential of the feedback. Besides the control effect that such feedback is going to have, the particular modality of use designed in the PP implemented in the teaching experiment of the UNISI Team can be related to the achievement of the specific educational goal concerning the development of the mathematical meaning of equivalence between expressions.

## The ITD pedagogical plan and its results

The alien PP is underpinned by the Activity Theory perspective.
The model of the activity highlights three mutual relationships involved in every activity: subjectobject, subject-community, community-object. Each of these relationships is mediated by a third entity. The relationship subject-object is mediated by artefacts, that both enable and constrain the subject's action. The relationship subject-community is mediated by rules (explicit or implicit norms, conventions regulating social interactions). The relationship community-object is mediated by the division of labour (different roles characterizing labour organization).

The structure of the tasks proposed in this PP can be described by the Activity Theory model in two steps: a first activity performed in paper and pencil environment is followed by a second activity
performed in Aplusix. If a contradiction emerges between the task performed in the paper and pencil environment and the task performed in Aplusix the activity can evolve (see figure below). The structure of the activities is specifically designed to allow possible contradiction emerge.

The feedback in Aplusix is a crucial element to highlight any contradiction between what is produces in paper and pencil environment and what is produced in Aplusix, and consequently it is a crucial element to fuel classroom discussion. Any contradiction can be explained and justified through a discussion with the teacher and the schoolmates. The achievement of the educational goals is expected as outcome of this interaction.


The type of tasks proposed in the PP are the following:
Convert a tree representation into a standard representation with paper and pencil and then verify the solution in Aplusix.

Convert a standard representation into a tree representation with paper and pencil and then verify the solution in Aplusix

Complete tree representations in paper and pencil and verify the solution in Aplusix

These tasks were designed to make emerge contradictions between the answers produced by students in paper and pencil environment and their answers given through Aplusix. The students used feedback to validate their answers and to understand the performed mistake. Discussion had an important role because it was orchestrated starting from students' answers with the aim of reflecting about the structure of numerical expression. For example discussion of task 1 and task 2 allowed teacher to orient students to reflect about the use of parentheses and the priority of operations.

The analysis of students' solutions has highlighted that, opposite to our expectations, the second task was easier respect to the first one. The difficulties emerged in the second task mainly depend on the poor experience of students in the tree construction.

On the contrary, in the students' solutions of the first task we have found many mistakes that are not present in those of the second task. These mistakes depend on the use of parentheses: many students wrote the standard representation without using them, even when they were necessary.

To explain this fact, a first consideration is that when students have to translate a tree representation into a standard representation have to choose if insert parentheses or not, while when he has to construct a tree starting from a standard expression they have to translate parentheses but not insert them in the tree.

A deeper analysis highlights that to accomplish the second task, students have to know the syntactical structure of the tree (how to build a tree) and they have to respect some computational rules. Students have to build the tree taking into account that collapsing bottom-up the tree, they will find the sequence of computation described by the standard expression. This task strengthens procedural skills, or in other words the "superficial structure" of numerical expression.

Opposite, to accomplish the first task procedural skills are not sufficient. Students have to interpret the tree structure.

Consider the following tree:


If a student read the tree in procedural way, he could be wrong in choosing between these three expressions: $\mathrm{a}+\mathrm{b}^{*}(\mathrm{c}+\mathrm{d})$ or $\mathrm{a}+\left(\mathrm{b}^{*}(\mathrm{c}+\mathrm{d})\right)$ or $\mathrm{a}+\mathrm{b}^{*} \mathrm{c}+\mathrm{d}$. In order to convert the tree in linear form parentheses must be inserted in the correct place, it is important to read the tree interpreting its systemic structure and this entails the capability to manage the numerical expressions in a structural way.

Task 3 was designed following the Activity theory model too. Students had to complete tree representations as shown the figure below.

This task is quite unusual in the experience with Aplusix. It was an interesting task because to solve it students had to focus the attention on structural aspects of an numeric expression. Students had to interpret the representations assigned with the task and to compare among them. Through their comparison students received hints that oriented them to focus the attention on structural aspect of the numerical expression to replace the question mark. In this solution the feedback was crucial. During the discussion students were invited to justify why in some cases contradictions are emerged.

However, opposite to our expectation, a lot of students solved this activity without difficulties. This fact may be explained as follows:

- Aplusix feedback provided students hints that oriented them in replacing the question mark
- To solve this task students had to focus the attention on structural aspects of a numerical expression


In summary, we can say that the modalities of employment of Aplusix and in particular of the feedback of Aplusix, designed according to the Activity theory, have been effective to achieve the educational goal. In general, results obtained by the comparison between an initial and a final test given to students respectively at the beginning and at the end of the experiment, has highlighted that students' achievements are consistent with what we envisaged a priori.

## I.6. Summary and contribution to the theoretical landscape

This summary concerns the modalities of employment of feedback as it has been exploited in the three different PPs, according the three different theoretical approaches and the common educational goal.

All the PPs selected the feedback in Training mode, and the modalities of use are expressed in different tasks proposed to the students. The different tasks were basically inspired by the common hypothesis drawn from Duval's theoretical approach that suggests, beyond the use of treatment tasks, the use of conversion tasks. Nevertheless, the general organization of these tasks differs greatly.

As far as UJF is concerned, the control provided by Aplusix prevents errors and supports adequate procedures. Thus, accordingly with Anthropological approach, this use of the feedback is expected to foster the development of suitable technique.

As far as the ITD is concerned, the dialectics between paper and pencil environment and Aplusix environment is fed by the control mode that, accordingly with Activity Theory, makes contradiction
emerge. The semiotic function of conversion tasks is exploited through the use of Aplusix. When the solution of the conversion is checked, the feedback provided by the DDA may highlight an inconsistency that requires to be explained.

As far as UNISI is concerned, the semiotic potential of the feedback is exploited. The tasks were designed to exploit the semiotic potential of the feedback signs; tasks explicitly asking the interpretation of the feedback signs - appearing step by step when the student operates on the expressions - are designed in order to make personal meanings emerge. Then, collective discussions are orchestrated by the teacher to make these meanings develop towards the mathematical meaning of equivalence.

What seems interesting is the fact that the different modalities of use of the feedback component seem not immediately in contrast, rather they highlight complementarity in respect of different possible aims concerning algebraic calculation and, specifically, equivalence. The controlled modalities (controlled tree) used by UJF, lead students to develop suitable techniques, that are not explicitly addressed for instance in the UNISI PP. The use of the controlled modalities (feedback) used to validate students' solutions used by ITD, leads students to grasp the meaning of equivalence between expressions facing contradictions and possibly overcoming them. The meaning of equivalence is directly addressed by UNISI PP where the formulation of the equivalence between algebraic expressions constitutes one of the basic aims. The link between feedback and equivalence, implicitly constructed through the ITD PP is explicitly developed in the UNISI PP through a dialectics between individual and

the set of potentialities of a specific component of a DDA.

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## II. Alnuset cross-case analysis

In order to be able to compare design choices made in the two pedagogical scenarios, didactical functionalities considered by the two teams and the outcomes of the experiments, the analysis reported in this document focuses only on parts of the experiments that have a same educational goal, namely the construction of meaning for the notions of equation, identity, truth value of an equation, truth set of an equation, equivalent equations.

## II.1. Identification

The teams involved are ITD-CNR Genova (Italy) and MeTAH Grenoble (France).

## II.2. Contextual elements

## ITD-CNR experimentation

## Local situational context

The experimentation activity, lasting 1 h 40 , has involved a class of 15-16 year-old students (Grade 10) attending a Classic Lyceum. The students worked in pairs using Alnuset. Previously, they had carried out 6 activities with Alnuset centered on notions concerning algebraic expressions. The whole teaching experiment lasted about 20 hours. The activity considered in this cross-case experimentation is centered on solving a $2^{\text {nd }}$ degree equation. In the previous school year, students had learnt to solve $1^{\text {st }}$ degree equations through symbolic manipulation.

## Relationship between research team, teacher and school

Some collaborations between the ITD team and the teacher involved in the experimentation were developed in the previous years. These collaborations were not continuous along the time and they were contextualized inside the standard mathematical curricula. Instead, as far as the present experimentation with Alnuset concerns, we asked the teacher to deeply modify the mathematical curriculum corresponding to the class involved. As a matter of fact, the mathematical curriculum of this class did not foresee solving second degree equations. Moreover, while in the algebra curriculum solving second degree equations is faced through the introduction of the well known formula, in this experimentation an alternative approach has been adopted.

As far as how research interventions were perceived by the educational system concerns, we must underline that the Liceo Doria, the school involved in the experimentation, is one of the most prestigious schools in Genoa. This has influenced in positive way the perception of the Principal about the research perspective.

## MeTAH experimentation

## Local situational context

The French experiment took place in a private senior high school in Grenoble, in a Grade 10 class with 34 students ( $15-16$ years old), during two sessions lasting 3 hours altogether, held in a computer lab where students worked in pairs on a computer.

Relationship between research team, teacher and school

The teacher of the class where the experiment has been implemented was a former colleague of one of the research team members, which facilitated the implementation and the organization of the experiment. The design of activities was negotiated with the teacher in order to integrate the experiment into the teacher's planned pedagogical sequence.

The Table 1 below synthesizes the contextual elements of both experiments:

|  | ITD-CNR | MeTAH |
| :--- | :--- | :--- |
| Kind of school | Classic Lyceum | Private high school |
| School level | Grade 10 (15-16 years old) |  |
| Number of students | 20h (whole experiment) <br> 1h40 reported activities |  |
| Number of hours | Computer lab with 2 students per computer |  |
| Classroom organization (whole experiment) |  |  |
| Relationship between <br> research team, teacher <br> and school | The teacher has collaborated with ITD <br> team in previous years but not in <br> continuous way along the time | The teacher was a former <br> colleague of one member of the <br> research team |
| Integration of <br> experiment the | The experimentation with Alnuset <br> required teacher to deeply modify the <br> mathematical curriculum corresponding <br> to the class involved. The <br> experimentation activities negotiated <br> with the teacher. | Experimental <br> negotiated with the teacher in <br> order to integrate the experiment <br> into her planned teaching <br> sequence |

Table 1. Contextual elements of experimentations with Alnuset.

## II.3. Theoretical frames

In this cross-case analysis the two teams share the same epistemological analysis related to the mathematical knowledge to be learned and to difficulties emerging in its leaning. Instead, different theoretical frameworks guided the two teams in the analysis of the whole process of mathematics teaching and learning, and of the mediation provided by the tool used in the didactical practice. The mathematical knowledge involved in their experimentations concerns the notion of algebraic equality. We present the main theoretical commonalities and differences that characterize the research of the two teams in the development of this notion in school practice. In doing this, we will try to show how the theoretical background of the two teams affects both the design of the didactical scenario centred on the use of Alnuset and the analysis of the teaching and learning processes related to its implementation in the school context.

## Epistemological analysis of the knowledge involved in the two experiments

The two teams share the assumption that important conceptual developments are needed to pass from numerical expressions and arithmetic propositions to literal expressions and elementary algebra propositions. As a matter of fact, in arithmetic only numbers and symbols of operations are used and the control of what expressions and propositions denote can be realized through some simple computations. In elementary algebra, instead, letters are used to denote numbers in indeterminate way and new conceptualisations are necessary to maintain an operative, semantic and structural control on what expressions and propositions denote (Drouhard 1995; Arzarello et al.
2002). The necessity of this conceptual development emerges clearly with the construction of the notion of algebraic equality. On the morphological plan, equality is a writing composed by two expressions or by an expression and a number connected by the " $=$ " sign. On the semantic plan, equality denotes a truth value (true/false) related to the statement of a comparison. When the expression(s) composing the equality is (are) strictly numerical, it is easy verifying its truth value through some simple calculations (e.g., $2 * 3+2=8$ is true while $2 * 3+2=9$ is false). Experiences with numerical equality contribute to structure a sense of computational result for the " $=$ " sign. This sense can be an obstacle in the conceptualisation of algebraic equality as relation between two terms, as highlighted by several researches (Kieran 1989, Filloy et al. 2000). When the expression(s) composing the equality is (are) literal the equality can present different senses because the value assumed by the letter can condition differently its truth value. In these cases the " $=$ " sign should suggest to verify numerical conditions of the variable for which its two terms are equal. There are cases where the two terms could never be equal whatever the value of the letter is, as in $2(x+3)=4 x-2(x-1)$. In other cases to interpret equality on the semantic plane, it is necessary to distinguish if it has to be considered as equation or as identity. The " $=$ " sign assigns to the equality the sense of equation when its two members are equal only for specific values of the letter. For example, the equality $2 \mathrm{x}-5=\mathrm{x}-1$ is true only for $\mathrm{x}=4$ and it is false for all other values. Instead, the " $=$ " sign gives to the equality the sense of identity when its two members are equal whatever the numerical value of the letter is, as in $2 x+1=x+(x+1)$. In order to master algebraic equality, a conceptual development of notions of equation, identity, truth value, truth set and equivalent equation is necessary. Moreover, to express the way in which a letter can condition the truth value of an equality, it is necessary to develop a capability to use universal and existential quantifiers, even though in implicit way.

Traditionally, conceptual construction of algebraic equality is pursued through solving equations using techniques of symbolic manipulation. Empirical evidence and results of research have highlighted that in many cases this approach does not favour a construction of an appropriate sense either for the notion of algebraic equality or for that of solution of equation. In more recent years, a functional approach to algebra has been introduced within the didactical practice allowing to articulate algebraic and graphical registers of representations. Even in this approach difficulties emerge.

The two teams share the hypothesis that the operative and representative opportunities of Alnuset can be effectively used to mediate the conceptual development necessary to master the notion of algebraic equality. Moreover, the two teams share that the operative and representative opportunities offered by this tool can emerge only within specific didactical practices. The two teams use different theoretical approach to design didactical scenarios to be implemented in classroom and to interpret the mediation offered by the tool used.

## Theoretical frameworks at the basis of the Italian teaching experiment

The Italian team uses the theoretical framework of the Activity Theory (AT) to design didactical scenarios and to analyze teaching/learning processes that take place in school context.

In the framework of the AT, every teaching/learning activity can be seen as an activity oriented towards a scholastic object (solution of a task, class discussion on a specific issue...) involving students, teacher and artifact to produce an outcome, namely the students' acquisition of a specific knowledge or ability (didactical goals). A specific model elaborated by Cole and Engeström (1991) in the framework of AT (cf. Figure 1) is particularly appropriate to study the relationships that take place in this type of activity.

This model highlights three mutual relationships involved in every activity, namely the relationships between subject and object, between subject and community, and between community and object. Each of these relationships is mediated by a third entity. The relationship between subject and object is mediated by artefacts that both enable and constrain the subject's action. The relationship between subject and community is mediated by rules (explicit or implicit norms, conventions and social interactions), while that between community and object is mediated by the division of labour (different roles characterizing labour organization).

The artefacts used in the activity mediate not only the relationship between the subject and the object but also that between the subject and the community and that between the community and the object.


Figure 1. Cole and Engeström' model of activity.
We have used this theoretical framework both to model the design of didactical scenarios mediated by Alnuset to pursue the didactical goals concerning the algebraic equality and to analyze the teaching and learning activity that took place during its experimentation in the class.

According to the Cole's and Engeström's model, in the algebraic activity the relationships between subject and community is mediated by algebraic rules that allow participants to define what can be considered as acceptable practice in that domain. Apart from the role of mediating element, in an educational perspective rules can be a didactical goal too. The transformation of the algebraic rule from being an individual-community mediator to an object of learning can take place only through a network of activities where shift of focus and breakdown occur.

Coherently with the AT framework, the ITD-CNR scenario has been designed in order to provoke contradictions and breakdowns in the activity and to promote a shift of focus related to the algebraic rule with the aim to determine a deep change and transformation in the object of the activity.

Having this aim in mind, specific tasks have been designed taking into account the following two didactical strategies:
a) comparing student's task solution obtained with pen and paper with the solution based on the use of Alnuset,
b) comparing students' task solutions and their interpretations of the representative phenomena occurred with Alnuset.

The use of Alnuset according to these two pedagogical strategies can have an important role to favour the transformation of the algebraic rule from being an individual-community mediator to an object of learning.

For example, specific tasks have been designed to exploit Alnuset for provoking a breakdown related to the use of specific rules between what a student has anticipated and what he has actually accomplished with the system; other task have been designed for provoking contradictions among the participants' interpretations of the representative phenomena mediated by the system connected to specific algebraic rule. As a consequence of these contradictions and breakdowns a shift of focus in the purpose of an action connected to the algebraic rule can emerge in the activity. When a breakdown and a shift of focus related to an algebraic rule occurs, a rule ceases to be a semiotic element that automatically mediate the individual action and his relationship with the community, and becomes the object of her/his target action.

The breakdowns and shift of focus that emerge in the transformation of the algebraic rule from being an individual-community mediator to an object of learning are mainly of semiotic nature. ITD team uses the Peirce's semiotic to explain how the operative and representative opportunities of Alnuset can favour the semiosis processes that characterize the described transformation.

Peirce describes a sign as a triad: a material sign which denotes an object of thought and an interpretant (which is another material representation of the relation between first material sign and the object). Moreover, Peirce distinguishes among three kinds of signs, namely indices, icons and symbols, according to the relationship that a sign establishes with its referential object. In the Peirce's framework the notion of rule (or general law or convention) is strictly linked to the notion of symbol: "a Symbol is a sign which refers to the Object that it denotes by virtue of a law, usually an association of general ideas, which operate to cause the Symbol to be interpreted as referring to that Object" (Peirce, 2003-CP 2.249). Peirce states that behind a rule or a convention of a sign there are always indexical and iconic links with the referential object and with its properties that can emerge through its interpretants. An icon is a sign that denotes its object by virtue of a quality that it shares with its objects; an index as a sign that denotes its object by virtue of an existential and physical connection that it has with its object.

ITD team thinks it is possible to exploit the operative and representative opportunities of Alnuset to arise breakdowns and shift of focus in the teaching and learning activity with the aim to allow students to grasp the indexical and iconic relationships behind the rules of algebraic symbols.

Let us consider, as examples, some operative and representative characteristics of Alnuset. We note that in Alnuset:

1. A variable is a mobile point on the line and an expression is a point on the line which depends on the value assumed by the variable. These points highlight an indexical relationship with their referential objects (numbers on the line) through the drag of the variable point.

The presence of two expressions in a post-it associated to a point on the line may mean:

- A conditioned equality, if taking place at least for one value of the variable during its drag along the line.
- A relationship of equivalence, if taking place for all values assumed by the variable when it is dragged along the line.
- A relationship of equivalence with restrictions, if taking place for every value of the variable when it is dragged along the line, but for one or more values, for which one of the two expressions disappears from the post-it and from the line.

The way expressions are represented on the Algebraic Line of Alnuset can mediate the development of the control over the conditions that determine the equality between two expressions or their equivalence.
2. A proposition within the Algebraic Line environment of Alnuset has an indexical relationship with its truth value that emerges through the drag of the variable on the line. As a matter of fact, the truth value of the proposition determines the colour of a marker associated to the proposition (green means true, red means false) during the drag of the variable on the line. The numerical set represented in a formal set notation in a window of the Algebraic Line has an indexical relationship with its referential object (numerical elements of that set) that emerges through the drag of the variable on the line. In fact, belonging or not of a numerical value of a variable on the line to the formal set notation determines the colour of a marker associated to it (green means belonging, red means not belonging) during the drag of the variable.

We note that the accordance between the colour of the proposition-marker and the colour of the setmarker is a representative event that can be exploited to validate the constructed set as a truth set of the proposition.

The way propositions and numerical sets are represented can mediate:

- the development of a control over the conditions that determine the truth of an equality or the equivalence between two equalities;
- the construction of ideas for the hidden universal and existential quantifiers required to master the truth set of a proposition.

3. The way expressions and propositions are manipulated in the Algebraic Manipulator of Alnuset can mediate:

- the development of an operative control of the way how to use the rules of algebraic transformation. In the Algebraic Manipulator environment after a selection of a part of expression or proposition by the user, the system activate all the transformation rules of the interface that can be applied on the selection performed. The application of one of the these rules determine the re-writing of the expression or proposition according to the transformation applied. This features of the Algebraic Manipulator of Alnuset support the recognition of the iconic relationship, namely the recognition of a structural similarity of form, between rule of algebraic transformations and algebraic forms on which they can be applied. This is at the basis of the capability to symbolically manipulate algebraic expressions and propositions.
- the development of semantic control as to what is preserved through their transformation. The result of the transformation can be automatically represented in the Algebraic Line environment and the representative events emerging in this environment can be effectively exploited as indices of the preservation of the numerical equivalence through the transformation.
- the development of a theoretical control of the way how to justify a new algebraic rule of transformation In this Manipulator the available rules are open-ended, in the sense that a new rule can be automatically created once it has been demonstrated. The new created rule can been considered as establishing an indexical relationship with its referential object, namely a theorem.


## MeTAH experimentation

Based on a preliminary analysis of Alnuset from utility, usability and acceptability points of view (Tricot et al. 2003), which brought to light main functionalities supposed to enhance learning of functions, equations and inequations, notions at the core of the Grade 10 math curriculum, MeTAH team decided to conceive a pedagogical scenario addressing these notions in Alnuset environment. In this report, we focus only on the notion of equation dealt with in both Italian and French experiments.

Three main theoretical frameworks have been chosen to underpin the design, implementation and analysis of the French experiment: semiotic registers of representation approach (Duval 1993, 1995), anthropological theory of didactics (Chevallard 1999) and instrumental approach (Rabardel 1995). Moreover, epistemological and cognitive considerations related to the mathematical notions at stake have also been taken into account. These considerations are shared with the ITD-CNR team and are presented above (cf. p. 2). In what follows, we explicit the way these theoretical frameworks and considerations were used and what choices they underpin.

Semiotic registers of representations

This approach was used at a very general level to analyze mathematical objects, their representations and manipulations available in Alnuset. Referring to this approach was motivated by the fact that mathematical objects are only accessible by means of their representations. According to Duval (1993), in order to be able to distinguish between an object and its representation, there is a necessity to have at one's disposal at least two different representations of the given object. The coordination of at least two registers of representation is also necessary to understand conceptual aspects that characterize the object. This coordination manifests itself by a capability to recognize if two different representations are representations of the same object. According to Duval (1995), semiotic registers of representation enable three kinds of cognitive activities: (1) formation of representations complying with the rules of formation of signs proper to the register (ex. $2 \mathrm{x}^{2}-1$ complies with the rules of formation within the algebraic register, $5 \sqrt{ }+$ does not); (2) treatment of a representation within a given register respecting transformation rules proper to the register (ex. in algebraic register, the rule $\mathrm{k}(\mathrm{a}+\mathrm{b})=\mathrm{ka}+\mathrm{kb}$ can be used to transform $3\left(\mathrm{x}^{2}+1\right)$ in $3 x^{2}+3$ ); (3) conversion of a representation in one register into another register (ex. the expression $3\left(x^{2}+1\right)$ in algebraic register is converted into "a product of 3 by a sum of a square of $x$ and 1 " in natural language register).

Let us look now at Alnuset through the lens of the semiotic registers of representation approach. Alnuset offers three different registers of representation of algebraic expressions:

- algebraic symbolic register in Algebraic Line and Algebraic Manipulator components. This is the usual register where representations of algebraic expressions consist of numbers, letters, operation symbols, relation symbols...
- dynamic graphical register in Algebraic Line component. This is a new register specific to Alnuset, in which algebraic expressions are represented by points on a number line whose position depends on actual values of their variables, which are also represented as mobile points on the line. Dragging the variable points along the line makes their values change and consequently the points corresponding to the expressions involving these variables move on the line accordingly.
- graphical register in Cartesian Plane component in which an algebraic expression is represented by a curve in a 2 -dimensional coordinate system.

These three registers are interrelated. For example, an algebraic expression defined in algebraic register is automatically represented as a point on the number line (coordination between algebraic and dynamic graphical registers), or there is a possibility, on the user's demand, to show a curve representing an expression while keeping visible the number line with the variable point and the algebraic representation of the expression (coordination of all three registers).

Looking for possible activities in Alnuset from the point of view of Duval's approach, it turned out that :

- A formation activity is only possible in algebraic register: a user can enter an algebraic expression either in Algebraic Line component, it is then automatically represented by a
variable point on the number line, or in Algebraic Manipulator component, it is then represented in the usual algebraic representation.
- A treatment activity is only partially possible in algebraic register within Algebraic Manipulator component: the user selects a sub-expression of a given expression and chooses a transformation rule from a list of applicable rules to be applied on the selected subexpression. The system applies the rule and provides the transformed expression.
- No direct conversion activity for the user is possible with Alnuset. Conversions are performed by the system automatically or on demand. For example in the Algebraic Line component, a variable is converted automatically in a mobile point on the line, an expression is automatically converted in a point on the line whose position depends on its structure and on the value of its variables assumed on the line. The system converts algebraic representations into graphical ones on demand. Moreover the system automatically converts numerical sets defined through a graphical model on the line into formal set notation.

Semiotic registers of representation approach appeared as not very fruitful for the design of experimental activities since, as we mentioned above, Alnuset performs conversions between registers either automatically or on demand, therefore no conversion activity can be done by the user. As far as treatment activity is concerned, it is only possible in Algebraic Manipulator component, which we deliberately left aside for the experiment having in mind a very short time allotted to it. However, we saw a potential of Alnuset in establishing links between registers, and mainly the links between algebraic and graphical registers mediated by the dynamic graphical register.

Functional approach to equations and inequations allowing the articulation between algebraic and graphical registers is at the core of the Grade 10 mathematics curriculum. Anthropological theory of didactics was thus adopted to analyze mathematical organizations related to the notion of equation both in math curriculum and in Alnuset.

## Anthropological theory of didactics

Within the anthropological theory of didactics, a praxeological analysis (Chevallard 1992) of the notion of equation in two distinct institutions, "Grade 10 textbook" and "Alnuset", allowed identifying types of tasks viable in both institutions and comparing techniques available in these institutions (cf. table 2). This analysis shows that while "Grade 10 textbook" techniques are based on algebraic transformations of algebraic expressions, Alnuset instrumented techniques rely on visual observations of expressions (their position on the algebraic line, colour feedback...), and (almost) no algebraic treatment is needed when applying these techniques (Krotoff 2008). Thus, Alnuset was considered as an appropriate tool to help students develop conceptual understanding of the notion of equation, without adding difficulties linked to algebraic treatment that many students do not master well enough. Consequently, types of tasks viable in both institutions were designed in a way to be solved with Alnuset.

| Type of task | "Grade 10 textbook" technique | "Alnuset" technique ${ }^{3}$ |
| :---: | :---: | :---: |
| $\mathrm{T}_{1}$ : Determine if two expressions are equivalent | $\mathrm{T}_{11}$ : by successive algebraic transformations of one expression check the possibility to obtain the other expression <br> $\mathrm{T}_{12}$ : by successive algebraic transformations applied on both expressions check the possibility to obtain identical expressions <br> $\mathrm{T}_{13}$ : check if graphs of the expressions coincide | $\mathrm{T}^{11}$ : enter the two expressions in Algebraic line environment and check whether the corresponding points coincide when $x$ moves on the line <br> T' ${ }_{13}$ : show graphs of expressions in Cart. Plane and observe if they overlap |
| $T_{2}$ : Solve an equation of the type $\mathrm{f}(\mathrm{x})=0$ | $\mathrm{T}_{21}$ : solve algebraically the equation (i.e. by means of algebraic transformations) <br> $\mathrm{T}_{23}$ : solutions are abscissas of intersection points between the curve representing $f$ and the $x$ axis | $\mathrm{T}^{\prime}{ }_{21}$ : in Algebraic line environment, use $\mathrm{E}=0$ command <br> $\mathrm{T}^{\prime}{ }_{22}$ : in Algebraic line environment, drag $x$ in a way to make $f(x)$ coincide with 0 , read the corresponding value(s) of $x$ (usable only when the solutions are integer) <br> $\mathrm{T}^{\prime}{ }_{23}$ : in Cartesian Plane show the curve of $f$, drag $x$ to make the mobile point on the curve coincide with the point(s) of intersection between the curve and the $x$-axis, read the value(s) of $x$ |
| $T_{3}$ : Solve an equation of the type $f(x)=g(x)$ ( g can be a constant) | $\mathrm{T}_{31}$ : solve algebraically the equation (i.e. by means of algebraic transformations) <br> $\mathrm{T}_{33}$ : utiliser la courbe représentative de fet de $g$ et lire les abscisses de leurs points d'intersection | $\mathrm{T}^{\prime}{ }_{31}$ : in Algebraic Line enter the expression $f(x)-g(x)$ using algebraic transformations, then use T' ${ }_{21}$ with $E=f(x)-g(x)$ <br> $\mathrm{T}^{\prime}{ }_{32}$ : in Algebraic line environment, drag $x$ and search for values of $x$ for which $f(x)$ and $g(x)$ coincide <br> $\mathrm{T}^{\prime}{ }_{33}$ : in Cartesian Plane, show curves of f and g, drag $x$ to make the mobile point coincide with the point(s) of intersection between the two curves, read the value(s) of $x$ |

Table 2. Praxeological analysis of the notion of equation in two institutions.
The instrumental approach (Rabardel 1995) was used in the analysis of students' activity wit Alnuset. We were focusing on the way the students get familiarized with the available functionalities of the tool, how they use them to solve the given tasks, how they develop the instrumented techniques and how they connect them with the mathematical knowledge at stake. In this way, we wished to identify to what extent Alnuset and the instrumented techniques help the students conceptualize the mathematical notions aimed at by the activities.

## II.4. Comparison of didactical functionalities

As was mentioned in the section III, both teams share the epistemological considerations related to the notion of algebraic equality, which led them to set up very similar educational goals. As a matter of fact, ITD-CNR team aimed at investigating the way Alnuset can mediate the notions of conditioned equality, solution of an equation, equivalent equations, truth value of an equality and truth set of an equation, and MeTAH team aimed at helping students construct meaning of the

[^2]notion of equation as a statement that is true for some values of the letter involved and of a solution of an equation as a value of the letter for which the equality is true. The two teams made use of the same characteristics and features of the tool, in particular the drag of variable points on the algebraic line, post-its associated to them and the $\mathrm{E}=0$ command. However, they differed in ways the tool was used to achieve the educational goals. These differences in the modalities of employment of Alnuset seem to result from different theoretical frameworks used to design experimental activities.

ITD-CNR team drew on the activity theory framework to design experimental activities. As was mentioned in the section III, coherently with this AT framework, the activities have been designed in order to provoke contradictions and breakdowns in the activity and in this way to promote a shift of focus related to the algebraic rule with the aim of favouring the development of a control on algebraic equality at a symbolic level. Specific tasks have been designed taking into account two different didactical strategies: comparing students' solutions obtained with pen and paper with the solutions based on the use of Alnuset, and comparing students' solutions and their interpretations of the representative phenomena occurred with Alnuset.

MeTAH team chose anthropological theory of didactics to frame the design of experimental tasks. A praxeological analysis of the notion of equation was performed in terms of types of tasks and techniques present in two institutions: "Grade 10 textbook" and "Alnuset". Moreover, bound by institutional constraints (the teacher of the experimental class required the tasks being close to the traditional ones so that they would easily integrate into her usual pedagogical sequence, which was in deep contrast with the ITD-CNR experimental conditions), we only looked at the types of tasks viable in both institutions. Significant differences were observed at the level of techniques associated to these tasks in the two institutions, which led us to question possible contribution of the instrumented techniques in Alnuset to the conceptualization of the notions at stake and to the development of traditional "Grade 10 textbook" techniques.

The following table presents the main aspects of the cross-case analysis of didactical functionalities considered by ITD and MeTAH teams regarding their experimentations with Alnuset.

|  | ITD-CNR | MeTAH |
| :--- | :--- | :--- |
| Educational <br> goals | Mediating the notions of conditioned <br> equality, solution of an equation, <br> equivalent equations, truth value of an <br> equality and truth set of an equation | Help students construct meaning of the <br> notions of equation as a statement that is <br> true for some values of the letter involved, <br> and of a solution of an equation as a value <br> of the letter for which the equality is true. |
| Characteristics <br> and features of <br> Alnuset | Algebraic Line: Drag of the variable <br> point, Post-it function, E=0 command, <br> Coloured marker associated to <br> proposition and numerical sets | Algebraic Line: Drag of the variable point, <br> pracking functionality, Post-it function, E=0 <br> command <br> Cartesian Plane: Showing graph of an |


|  | Algebraic Manipulator: transformation <br> of propositions by means of the <br> available rules, possibility to send the <br> transformed form of an expression or <br> of a proposition to Algebraic Line |  |
| :--- | :--- | :--- | :--- |
| Modalities <br> use | Tasks designed according to the AT <br> and taking into account the two <br> didactical strategies described in <br> section III. | - Students working 2 per computer <br> - Alternation of pairs work and whole class <br> discussions |
| -Traditional school algebra tasks designed <br> to be solved by using Alnuset <br> instrumented techniques |  |  |

Table 3. Didactical functionalities of Alnuset considered by the two teams.

## II.5. Results of the cross-case analysis together with illustrative examples

The two teams have used different frames to analyze the outcomes of their respective experiments. ITD-CNR team used Peirce's semiotic approach to highlight the way Alnuset mediates the relationships of the signs used in algebra with their referential objects, while MeTAH focused on the instrumented techniques developed by the students in interaction with Alnuset and on the discourse about these techniques in order to analyze whether and to what extent they contribute to conceptual development of mathematical notions at stake and how they can be related to the traditional school techniques.

## Italian experimentation

Let us remind that the Italian team was involved in the design and implementation of Alnuset. For this reason the ITD team's research goal was to verify if the use of Alnuset in educational activities can support the conceptual development of some algebraic notions (such as the notions of equation as conditioned equality, of solution of an equation, of equivalent equations, of truth value of an equality and of truth set of an equation).

In this report we will focus on some examples to illustrate how Alnuset was used to developed a conceptual understanding of the " $=$ " sign in algebraic domain (that is to say as sign between algebraic expressions) in order to conceptualize, in particular, the notion of equation as equality conditioned by some values of the variable.

To master these notions in the algebraic activity, it is necessary to use signs such as letters, operational signs, numbers which are used to compose others signs such as algebraic variables, expressions and propositions. Moreover, to master these signs on the operative level, it is necessary to recognize the relationships of these signs with their referential objects (numbers, sets of numbers, truth values), namely it is necessary to develop the capability to practice a semantic control over them. To explain how Alnuset can mediate these relationships (between variables, expressions, and propositions and their referential objects), we referred to some elements of the Peirce's semiotic frame (Peirce, 2003).

Our experiment activity is composed by several tasks. The first task aims at allowing students to explicit their own conception of the algebraic equality notion.

Task: Consider the following two polynomials: $x^{2}+2 ; 2 x+3$. Explain what it means putting the equal sign between them, or, in other words, how you interpret the following writing $x^{2}+2=2 x+3$.

Many students attribute to the " $=$ " sign the meaning of computation result, despite they were already faced with $1^{\text {st }}$ degree equations. A typical students' answer is: "To put the equal sign between two polynomial expressions means that these expressions have the same result". For many students inserting the equal sign between two expressions suggests the idea that the computation result of the two terms has to be equal when a value is assigned to the letter. In other words, the equivalence is verified through a numerical substitution.

In the following task students were asked to represent the two expressions on the algebraic line of Alnuset to verify their answers "Insert the polynomials $x^{2}+2 ; 2 x+3$ in the algebraic line of Alnuset and verify your hypothesis". Accordingly to the idea that tasks must provoke contradictions and breakdowns in the activity thus promoting a shift of focus related to the meaning of the equal sign in algebra domain (with the aim to favour the development of a control on algebraic equality at a symbolic level), students are asked compare their tasks solutions obtained with pen and paper
 that based on the use of Alnuset.

Dragging the mobile point x along the line, and observing that the points corresponding to the two expressions $x^{2}+2$ and $2 x+3$ move accordingly, all students observed that there are only two values of x for which the points corresponding to the two expressions are close to each other, almost coincident. All students used the "start tracking" function which allowed them to visualize at the same time on the screen the two expressions which are far from each other when the value of x moves along the line. We can observe that in the algebraic line of Alnuset, the indexical relationship between the points corresponding to the expressions $x^{2}+2 ; 2 x+3$ and their objects of reference (the numerical values assumed by the two expressions for the different values assumed by the variable x involved) was made explicit. This dynamic indexical relationship allowed students to grasp the meaning of functional dependence of an expression to a variable. Thus, through this exploration, students experienced that equality of these two expressions is conditioned by numerical values of the variable, which is crucial to develop the conditioned equality notion. In previous activities with Alnuset, students experienced that every point of the algebraic line is associated to a post-it that contains all expressions constructed by the user denoting that point. In order to verify equality of two expressions, the students tried to find values of $x$ for which the two expressions belong to the same post-it. We observe that the post-it can be interpreted as an index in the Peirce's sense. As a matter of fact, the post-it indicates that the expressions contained in it are equivalent for the value of the variable form that they depend on. Thus, for this value of $x$, the two expressions correspond to the same point and they indicate the same numerical value. In this way, the pos-it of Alnuset mediated the equivalent expressions meaning.

Since these values had to be constructed on the line because they were irrational (in full range domain, only integer numbers are points represented on the line by default), the students could not verify this directly: "we don't understand what is the number...it will be 2 point something...even if we use zoom in we don't understand ...". The technique mediated by Alnuset to find these irrational
numbers requires transforming the equation into its canonical form ( $x^{2}-2 x-1=0$ ), representing its associated polynomial on the line and using a specific command to find roots of this polynomial. Our hypothesis was that this technique could favor a conceptual development of notions of equivalent equations and of truth value/truth set of an equation. The transformation was realized in the Symbolic Manipulator. In order to transform the equation $x^{2}+2=2 x+3$ in its canonical form, students had to apply the rule $\mathrm{A}=\mathrm{B} \Leftrightarrow \mathrm{A}-\mathrm{B}=0$ to the equation. The selection of the equation allowed students to identify the algebraic structure which they had recognize in that of the rule $A=B \Leftrightarrow A$ $\mathrm{B}=0$. We can observe that the choice of a rule among those available in the interface is possible by recognizing iconic links between the algebraic structure of the rule and that of the part of expression or proposition selected (in our case, students have selected the whole equation).

In order to explore quantitative relations linking the two expressions $x^{2}+2$ and $2 x+3$ to the expressions $x^{2}-2 x-1$ and 0 , students had to understand that the values of $x$ for which $x^{2}+2$ was equal to $2 x+3$ were the same for which $x^{2}-2 x-1$ was equal to 0 . The didactical strategy used was to ask students to formulate an hypothesis of solution and then to validate it through the use of Alnuset. Contradictions could occur and, in that case, Alnuset could be used to explore and describe the correct relationship between $\mathrm{x}^{2}+2$ and $2 \mathrm{x}+3$ and $\mathrm{x}^{2}-2 \mathrm{x}-1$ and 0 .

Task: Make a hypothesis about the relationship among the three polynomials $x^{2}+2 ; 2 x+3 ; x^{2}-2 x-1$ imagining what you could observe if you represented them on the algebraic line and if you dragged $x$. Use algebraic line to verify your hypothesis.

A posteriori, we realized that the formulation of this task was misleading since it oriented the students to search for a relation among the three polynomials rather than between couples of terms of the two equations. Some students dragged the variable to explore if there were values of x for which the three polynomials could denote the same value on the line. They verified that such a value does not exist. Even if this exploration was not expected, it proved an important reference to overcome the following misconception, quite common in the students, concerning the equivalence of equations: two equations are equivalent if all their terms are equal for some values of the variable. A new formulation of the task by the experimenters allowed students to focus on couples of terms of the two equations. Exploiting the drag of the variable $x$ they experienced that, the values of $x$ for which the points corresponding to the expressions $x 2+2$ and $2 x+3$ are closer each other, are the same for which the point corresponding to the expression $\mathrm{x} 2-2 \mathrm{x}-1$ is close to the point 0 . Once again, the points corresponding to the expressions can be interpreted as indexical signs linking, in dynamic way, the expressions with the values that they assume dragging the variable from which they depends on. These indexical signs and their dynamic use, allowed students to assign meaning to the exploration on the Algebraic Line component. In others worlds, students understood that, in order to find values of x for which $\mathrm{x} 2+2$ is equal to $2 \mathrm{x}+3$, it is sufficient to find values of x for which $\mathrm{x} 2-2 \mathrm{x}-1$ is equal to 0 . Thus, they can begin to grasp the meaning of equivalent equations notion. Since these values are irrational, students had to construct them on the algebraic line. For this reason, students used the command $\mathrm{E}=0$ to find the irrational roots of the polynomial $\mathrm{x}^{2}-2 \mathrm{x}-1$ and to automatically represent them on the line. The technique to use the function $\mathrm{E}=0$ supports the construction of the meaning for the notion of roots of a polynomial. As a matter of fact, the student has to drag x to approximate the polynomial to 0 . When this is done, the system automatically produces the exact value of the root. The process of calculation of the polynomial root is
graphically represented by a triangle which changes progressively its color from red to green. The triangle is an index, in the Peirce's sense, of the polynomial root.

Through these tasks the idea of equivalent equations (equations having the same truth set) can begin to emerge. As a matter of fact, the post-it allowed students do verify that, the points corresponding to the expressions $x^{2}-2 x-1$ and 0 overlap (that is they refer to the same point) and, for the same values of $x$, the points corresponding to the expressions $x^{2}+2$ and $2 x+3$ overlap. This idea will be consolidated through the exploitation of a new dynamic feedback offered by the system. We note that in the algebraic line environment expressions are represented on the line while equalities are represented in a specific window named "sets" and they are associated to a marker (a little dot) whose color is managed automatically by the system. The marker is green if, for the current value of the variable on the line, the equality is true and, conversely, it is red if the equality is false. The color of the marker is an index in the Peirce's sense. The green color indices that for the current value of $x$ the equation is true, instead, the red color of the marker indices that the equation is false. The analysis of the results of the experimentation shows that some students have grasped the meaning of equivalent equation observing the accordance of color between the marker associated to the equation $x^{2}-2 x-1=0$ and that associated to the equation $x^{2}+2=2 x+3$, as shown in the following dialogue.

Student 1: When $x$ is $1-\sqrt{2}$ the two expressions are equal and these [dots] are green. So, since the solution of this equation is $1-\sqrt{2}$ then also for the other equation is the same.

Student 2: and for the other value $[1+\sqrt{2}]$ it is true the same
Student 1: yes, for these values the two equations are true


Others students, instead, have grasped this notion observing that, for the same value of $x$, the expressions $x^{2}-2 x-1$ and 0 belong to the same post it and the expressions $x^{2}+2$ and $2 x+3$ belong to the same post-it.

Student: If I drag $x$ on $1+\sqrt{2}$ and on $1-\sqrt{2}$, the expressions of the first equation belong to the same post-it, namely $x^{2}-2 x-1$ and 0 are coincident for these values of $x$. For the same values of $x$ even $x^{2}+2$ and $2 x+3$ belong to a same post-it.


In both cases we can observe that the indexical link between the sign (color of marker or post-it) and its respective object of reference (truth value of the equation, equivalence between expressions) has helped students to build the meaning of equivalent equation notion.

## French experimentation

In what follows, we provide examples of tasks designed in a way to make the students' strategies evolve from possibly using Alnuset only to verify solutions found algebraically to those based on instrumented exploratory techniques. A few results of the experimentation are provided illustrating how Alnuset was used to develop a conceptual understanding of the " $=$ " sign and some instrumental issues are discussed.

The first task involving equations consisted in finding solutions of $f(x)=4$, with $f(x)=x^{2}$, after having studied the function f with Alnuset. The task was intentionally quite simple: the students could either solve the equation algebraically and verify the result with Alnuset, or solve the equation with the tool by dragging x along the algebraic line and looking for values for which $\mathrm{x}^{2}$ coincides with 4 . Both strategies appeared to almost the same extent. However, students who used the exploration strategy to find solutions with Alnuset succeeded better than those who used the tool just to verify the results found by solving the equation algebraically, since these often provided only one, positive, solution. This example shows two different schemes of use of Alnuset emerging in the activity, thus transforming the artefact either to an instrument whose main function is to validate a conjecture, or to an instrument that can be used to explore a given situation and raise conjectures.

The task asking to solve the equation $x^{2}=3 x+4$ was proposed next to prompt students to use Alnuset technique of dragging $x$ on the line and searching for values for which the equality is true. Indeed, the students did not know yet algebraic techniques for solving such 2 nd degree equation. Using the Alnuset technique requires to make sense of the " $=$ " sign as meaning that the two expressions have the same value for some value of x , and thus also to distinguish between a letter standing for a variable and for an unknown. The students were first asked to determine whether $1,-1$ and 2 are solutions of the equation. This question was intended to reveal students' conceptions of the notion of solution of an equation. Almost all students succeeded the activity. However, the following dialogue between two students reveals the student's S1 conception of a solution linked to the arithmetic sense of the " $=$ " sign:

## S1: You have to find 1. No, $3 x+4$ must be equal to 1, the solution.

S2: No, you have to put $x$ on 1 and the... what do you call it [pointing at $3 \mathrm{x}+4$ ]... Because $x^{2}$ should be equal to ... the thing, equation and this isn't the case (Figure 2a).
$\mathrm{S} 1: \quad$ But it's the result this [pointing at 1].
Indeed, it seems that S1 considers a solution of an equation to be the "result" or the value of the expressions: if 1 is a solution of $x^{2}=3 x+4$, then $\left(x^{2}=\right) 3 x+4=1$. This conception emerged also when the students checked for -1 . The student S 2 grasped the targeted technique: "On the other hand, -1 is the solution since $f(-1)$ equals this equals this equals this" (Figure 2b), and explains it to S1: "To find the solutions, you drag $x$ until $x^{2}$ and $3 x+4$ overlap".


Figure 2. (a) 1 is not a solution since $x^{2}$ and $3 x+4$ do not overlap when $x$ is on 1 ; (b) -1 is the solution.

The students were then asked to find other solutions of the equation if there are any. This task was much more difficult for the students. Only half of the pairs succeeded it. The main obstacle was the fact that when $x=4$ (the other solution), the expressions $x^{2}$ and $3 x+4$ went out of the screen. The students did not spontaneously resort to using "tracking" functionality allowing to keep visualising the expressions taking bigger values, which the students had used previously. Teacher's intervention was necessary to remind the availability of this functionality, which helped the students to successfully finish the task. Such observations point to the issue of instrumental genesis in students, which can be a rather long-term process, especially in the case of innovative functionalities such as "tracking". During the whole class discussion, many students mentioned having used post-its and the "overlapping" strategy in Alnuset combined with verifying algebraically whether the two expressions $x^{2}$ and $3 x+4$ have a same value when $x$ is assigned a value found as a solution of the equation with Alnuset. This evidences about the articulation between the instrumented Alnuset and the traditional school techniques. Moreover, the teacher had often taken opportunity during the whole class discussions to address the issues of available functionalities and the ways of using them to solve the given tasks, which appeared crucial to students' progress both in solving mathematical tasks and mastering the tool.

## II.6. Potential offered for the theoretical landscape

This summary concerns the use of Alnuset and some of its characteristics and features according to different theoretical approaches (cf. figure 3). These different approaches led to different modes of use of the same functionalities of the tool to achieve the same educational goal (conceptual understanding of algebraic equality). As in the case of the cross-case analysis concerning Aplusix, we can conclude that the different theoretical approaches provide different perspectives to enrich the set of potentialities of the features of a DDA.


Figure 3. Different modalities of use of Alnuset driven by different theoretical approaches.

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## III. Casyopée cross-case analysis

## III.1. Identification

- Teams involved:
o DIDIREM (France): familiar DDA
o UNISI team (Italy): alien DDA


## III.2. Contextual elements

## School level:

- DIDIREM Grade 11, 1 group with 30 students and 1 group with 18 students
- UNISI Grade 13, 1 group


## Physical context:

Classroom equipped with computers, overhead projector.
Students work in pairs.
For UNISI: Students were sometimes involved in collective discussions

## Length of the scenario:

- DIDIREM: 11 hours
- UNISI: 12 hours


## Teachers:

The two teachers involved in the DIDIREM experiment took partly part to the design of the experiment In particular, they wrote the exercise sheet. The experiment was designed to be inserted in the curriculum of the class. It was well accepted by the institutional context of the school. It is worth to be mentioning that Casyopée was not a new DDA for these teachers but that have not experimented before the extension of Casyopée developed in ReMath and at the core of the crossexperimentation. The teachers in the UNISI experiment had no previous knowledge or experience with Casyopée.

## III.3. Theoretical frames

The UNISI team has mainly structured its pedagogical plan according to the Theory of Semiotic Mediation (Bartolini Bussi \& Mariotti, 2008). A basic assumption of the theory is that mathematical meanings are rooted in the action with the artefact, and developed through social interaction in classroom. Identifying the semiotic potential of an artefact is the starting point for developing a teaching/learning sequence that involves the use of such artefact. This means to identify the potential that an artefact has with respect to some mathematical meanings, in relation to the tasks in which it is used. This theoretical framework (TSM) inspired both the specification of the educational goals and the organization of the activities in iterative cycles. In particular the TSM led the UNISI team to devote attention towards the design of the teacher's action in the pedagogical plan. In fact, the teacher plays a crucial role throughout the whole pedagogical plan, especially for
fostering the evolution of students' personal meanings towards the targeted mathematical meanings and facilitating the students' consciousness-raising of those mathematical meanings.

The DIDIREM team splits its theoretical approach into several theoretical frames which shape their pedagogical plan: the Instrumental Approach (Artigue, 2002), the theory of Situation (Brousseau, 1997) and at last the theory of anthropologic didactic (Chevallard, 1999). The first frame aims to go further than a simple familiarization with the DDA and to help the students constructing a mathematical instrument. This process goes hand in hand with the learning process. The process is accurately designed through a careful choice of mathematical tasks, with an adidactical potential, whereas the definition of the teacher's actions and role escapes the design of the PP. Finally, the TAD is called upon to manage instrumental distance between institutional and instrumental knowledge.

## III.4. Comparison of the UNISI and DIDIREM approaches using the construct of didactical functionality

The two pedagogical plans, evidently share some characteristics but also have apparent deep differences. In this section we use the frame provided by the construct of Didactical Functionality to develop a more systematic comparison between the two pedagogical plans.

## Tool features

The two pedagogical plans are not only centred on the use of the same DDA, but also on the use of the same DDA features. In fact both exploit especially:

- features of the dynamic geometry environment: the commands for creating fixed, free or constrained points, for dragging free or bonded points, for creating points with parametric coordinates, and the corresponding feedbacks of the DDA;
- features of the geometric calculation environment: the commands for creating "geometric calculation" associating numbers to geometrical objects, for choosing (independent) variables, for creating function between the selected variable and calculation, and the corresponding feedbacks;
- features of the algebraic environment, including the commands for displaying and exploring graphs of functions, for creating and manipulating parameters, for manipulating the algebraic expressions of functions, and the corresponding feedbacks.


## Educational goals

Different educational goals are associated to the use of those features. More precisely, one can recognize that both pedagogical plans share a common focus on some mathematical notions: function (in particular, conceived as co-variation), variables (independent and dependent) and parameters. Moreover the two pedagogical plans present, among other tasks, two optimization problems sharing the same mathematical core (see sections...). But, besides those surface similarities, there are profound differences.

UNISI educational goals are to mediate and weave meanings, related to the notions of function, variable and parameter. With that respect the UNISI team assumes, on the one hand, that those
notions are familiar for students, and, on the other hand, that those notions are not elaborated in depth. Hence the UNISI pedagogical plan aims at helping students gain a deeper consciousness of the mathematical meanings at stake and re-appropriate them in the more global frame of modelling. In addition the UNISI objective includes the shared and decontextualized formulation of the different mathematical notions in focus.

The DIDIREM objectives are mainly to use potentialities of representations offered by Casyopée to introduce some new mathematical knowledge concerning quadratic polynomials and studies of optimization problems through functional modelling in the context of geometry. This knowledge has been chosen for two main reasons: its relevance with respect to the French curriculum and the interest to approach it through interaction between different systems of representations.

## Modalities of employment

In accordance with the different objectives and the different pedagogical cultures, the modalities of use are different as well.

The UNISI pedagogical plan has an iterative structure. Students' activity with Casyopée alternates with classroom discussions, and after each session the students are required to produce individual reports on the performed activities. This structure is meant to foster the students' generation of personal meanings linked to the use of the DDA and their evolution towards the targeted mathematical meanings together with the students' consciousness-raising of the mathematical meanings at stake. That process is constantly fuelled by the teacher, whose role is crucial. Accordingly the teacher's role is explicitly taken into account in the design of the pedagogical plan, which provides hints for his(her) possible actions. The tasks used are optimization problems set in a geometrical frame. Their solution and the reflection on these solutions are fundamental steps towards the achievement of the designed educational goals. Also the familiarization with the DDA has to be considered within that perspective: as already mentioned, it aims at making students observe and reflect upon the "effects" of their interaction with the DDA itself. $A d$ hoc tasks are designed for that purpose.

Instead of that, the DIDIREM team pays specific attention to a progressive use of the DDA combining artefactual and mathematical knowledge. Indeed, students work only in the algebraic window during session 1 , then only in the geometrical windows in session 2 ; finally section 3 gives an opportunity to reinvests the knowledge in the two environments. Moreover, all the tasks proposed are mathematical ones and are elaborated in order to allow students make progress alone working on the problem and to construct their new knowledge thanks to the feedbacks received through their interaction with the DDA.

## III.5. Results of the cross-case analysis together with illustrative examples

## Methodology of the cross analysis

The two teams used differently their theoretical frameworks in the design and in the analysis of the data of the experimentation. For the DIDIREM team, the design of the sequence was inspired by several theoretical frameworks, Instrumental Approach, TDS and ATD, whilst the analysis of the
data coming from the experiment was based on the TDS and Duval's theory of semiotic registers of representation. For UNISI, the theory of semiotic mediation both inspired the design and framed the analysis of the data. However, both teams used specific constructs within their own theoretical frames, as tools for the analysis. UNISI tries to identify semiotic chains collectively constructed during the discussion; that is to say sequences of signs, which show semiotic links with the artefact and the mathematics as well. Through such semiotic chains, it is possible to identify the development of signs: in fact, their meanings are rooted in the use of the artefact but they are expected to evolve, to be de-contextualized and clearly related to mathematics. DIDIREM, referring to TDS, tries to characterize the "structuration of the milieu" and of the didactic contract. In particular, each transcription is split into episodes and for each episode the researchers specify the different "milieu" involved (material milieu, evoked milieu...), the knowledge status (new, in progress or acquired) and who is responsible for the discussion (teacher or student). Both teams use these respective constructs in order to understand the progression of students' knowledge along the sessions, and possible reasons for that progression, taking into account the teachers' mediations.

In both cases, the way data are collected, analyzed and interpreted was shaped by the selected theoretical frameworks. Having in mind the search for possible complementarities between the theoretical frames used, it was decided to enrich the analysis by crossing the theoretical tools used on the same piece of data. This decision was challenging not only because of the intrinsic difficulty of analyzing data coming from an alien experiment, but also because in principle we could not be sure of the consistency of this cross analysis. The first challenge was the selection of the piece of data. The difficulty resided in the different modalities of use that did not provide the same kind of data. Moreover in the UNISI experimentation, a key element of the semiotic mediation process was made from the collective discussions, whilst in the French experimentation we could not find comparable phases. In fact the most similar situation, e.g. institutionalization phase or situation, does not present the same features in respect to the interplay of responsibilities between the actors students and teacher. To overcome this difficulty, it was decided to select two pieces of data from the two experiments that were important for both experimentations and presented features for the analysis relevant for both teams. These pieces of data are: one excerpt from the transcript of a collective discussion and one excerpt from the transcript of a laboratory session where there was evidence of collective interaction between students and the teacher.

The second challenge was the method of analysis: we had at least to choices. Carrying out the two analyses separately and then comparing, coordinating the use of the different theoretical tools in order to carry out the analysis. We chose the second option. We present below some illustrative elements of the analysis carried out and of their results.

## Some results of the cross analysis

Example 1: Cross analysis of the beginning of a collective discussion (session 4) from the UNISI experimentation

Teacher: "What are the issues in the resolution of this second type... I mean, basically what were you asked to do as homework, no? I mean what are the main issues when you try to solve this kind of problem, what kind of problem where we looking at... Come on, who wants to start? Come on guys. This is the one that... every problem is a different world right? Because every problem has its
own solution. What is the important thing that you need to do? You need to try to understand which generalities there are and then try to apply them to all the possible problems in the future with and without the software because these are also two important aspects and this... the software guy did you because it emphasizes particular important issues. Corinna would you like to try to start? Break the ice."

UNISI analysis: The teacher specifies the motive of the activity. She asks to go back to the task focusing on Casyopée: in order to make personal senses emerge in relation to the use of the DDA. She asks for generalization to promote evolution of personal senses towards mathematical meanings.

DIDIREM analysis: The teacher has the responsibility of this episode. She declares the objectives of the discussion: double de-contexualization from the task and from the artefact. She also prepares the milieu for the collective discussion. She evokes the "material" milieu, the milieus associated with the tasks and a fictitious milieu without DDA together with a generic problem. This is completely different from the DIDIREM objective. As a matter of fact, the appropriation of an artefact and even its transformation into a personal tool is an objective explicitly suggested by the instrumental approach favored by DIDIREM.

Example 2: Cross analysis of an excerpt, general tools
2. Cor: "[...] First of all we had to choose the triangle by giving coordinates" [Cor recalls the steps to represent the geometrical situation within Casyopée DGE]
5. Luc: "But you have to choose a mobile point, first [...]"
6. Teacher: "Does everybody agree? [...]How would you label this first part? [...]"
7. Students: "Setting up"
8. Teacher: "Luc has just highlighted something [...] do you see anything similar between the two problems?"
9. Sam: "One has always to take a free point which varies, in this case, the areas considered [...]"
10. Teacher: "Then we have a figure which is..."
11. Students: "Mobile."
12. Teacher: "Mobile, dynamical. Let us pass to the second phase. Andrea, which is the next phase? [...]"
13. And: "The observation of the figure would let us see... we need to study that figure and observe what the shift of the variable causes..."
14. Teacher: "Ok, then? Everybody did that, isn't it?"
15. Sil:" We computed the area of the triangle and of the parallelogram, we summed them, and by shifting the mobile point one observed as [the sum of the areas] varied [...]"

DIDIREM analysis: globally speaking, items 2-12 form an episode aiming at recalling the general characteristics of the situations 2 (first modelling problem) and 4 (second modelling problem). Change of the episode is marked by the teacher who has the responsibility of the episode (management and validation, "let us pass to the second phase"). More precisely Cor and Luc share
micro-responsibility in item 2 to 5 . Cor evokes the material milieu and recalls the actions performed. Luc completes Cor's description. Cor and Luc refer to the situation 4. Then in item 6, the teacher breaks the didactic micro-contract (Hersant \& Perrin-Glorian, 2005) by putting into question what has been said before because he wants to name the first step ("setting up").

UNISI Analysis: There is an emerging semiotic chain: "mobile point (5) -> free point (9) -> variable (13) ->mobile point (15)". Students recognize that geometrical objects can be treated as variables. Students are weaving different meanings related to the meaning of variable. In item 12, the teacher has a Semiotic action - request of naming - for promoting generalization and synthesis.

Example 3: Cross analysis of an excerpt: Topaze effect?

1. Prof.: "Uh, a specific button, come on what is it called?"
2. Students: "OM squared... OM square root..."
3. Prof.: "Can we find this button, this way of working that lets us do this first part?"
4. Students: "Geometric calculation."(English term used bye students)
5. Others: "Dynamic geometry."(English term used bye students)
6. Student in the first row with a white T-shirt: "No, Geometric calculation ... Dynamic geometry, what is it called, it's the one where there is the figure."
7. Prof.: " so if we want to call it something in Italian we could give it a name... Shall we write geometric calculation like she says? Do you all agree?"
8. Student: "In Italian."
9. Student: "I don't understand why in English."
10. Prof.: " Exactly. so, let's do this since we don't have the software in front of us anyhow. So let's find an Italian word that you think...
11. Student: "Calcolatore geometrico."

One may find a Topaze effect (Brousseau 1997, p.25) when the teacher wants to introduce a mathematical sign. It may happen that although the development of the meanings is mature for moving to the mathematical context, a gap has to be overcome. Actually students cannot know in advance what precisely the link with mathematics is, and to some extent they are invited to guess what the teacher wants them to say. But this time the objective is not to please the teacher per se. Neither the teacher nor the student feed the "illusion" typical of the Topaze effect: they share the objective of the intervention and assume the sense in relation to the common goal of relating artefact use and mathematics. For this reason the given answer is senseless, the sense being developed, worked on, and weaved together.

Example 4: Cross analysis of a dialogue during a laboratory session in the DIDIREM experimentation

The following dialogue has been extracted from a laboratory session (session 2). Students have a triple task during the episode:

- To adjust the parameters' values in an algebraic expression of a quadratic function such as the graph of this function become similar to a given graph in the graphical window of Casyopée;
- To explain the method used;
- To provide an interpretation of the parameters' roles.

This excerpt takes place when J (student) calls the teacher ( P ) to get help in order to interpret the parameters.

1. $\mathbf{J}:$ Monsieur dans les méthodes comment on peut mettre?
2. $\boldsymbol{L}: A h$
3. $\boldsymbol{J}:$ Parce que nous il n'y a pas vraiment de méthode
4. L : Oui c'est avec le ..
5. $\boldsymbol{P}$ : Votre méthode ça consistait à faire quoi finalement ?
6. $\boldsymbol{J}$ : ben on regardait le graphique par exemple là fallait là décaler donc on pouvait agir sur le paramètre
7. $\boldsymbol{P}$ : Mais la façon dont vous avez décalé, est-ce que vous avez pensé à quelque chose plus ou moins consciemment
8. J : Ben oui par exemple $k$
9. $\boldsymbol{P}:$ Vous pensiez quand même à des... quel type de transformation ?
10. J : Ben des translations
11. P : Bon des translations et le choix

DIDIREM Analysis: this episode takes place during a learning situation with a particular mesodidactical contract. Students have to understand links between their manipulations on Casyopée and the values of parameters in different algebraic forms. Inside this meso didactical contract, a micro contract is initiated by some students ( J in particular) where responsibility of knowledge is given back to the teacher. In this micro contract, the teacher injects knowledge in the milieu to allow the students to continue. The student wants to "explain the method" as it is requested, but to do this, he needs the mathematical vocabulary of transformations. Like in the previous Italian excerpt, the teacher's role is to help students to express mathematically the observation in the software. As a difference with the Italian, this excerpt is taken during a problem solving phase, with less control by the teacher than during a collective discussion. It is then possible that the students are driven towards a mathematically formulated method without enough reflection. Certainly the teacher is here in difficulty because the pedagogical plan (scenario) included the necessity of a mathematisation without planning an adequate period of collective elaboration.

UNISI Analysis: The use of the term "transformation" has not the immediate desired (?) effect of making J relate parameter k and the transformation. Instead, it feeds the construction of a semiotic chain from the artefact signs "décaler", etc... to the mathematical signs: "transformation", "translations", etc. In fact, the term "transformation" is immediately decoded by students as a mathematical term. Students' answer is in tune: they introduce the mathematical term "translation". Despite "translation" is a mathematical term, there is the risk that the meaning attached to it is still confined to the actual use of Casyopée. The teacher does not ask students any de-contextualization.

Example 5: analysis of a institutionalization episode

1. Pet est-ce que ça vous fait penser à quelque chose que vous connaissez depuis la troisième ça, à quoi ça sert de factoriser? Avec une phrase que vous récitiez comme une récitation. Qu'est ce que vous récitiez en $3^{\text {ìme }}$ ?
2. J (et d'autres) «Un produit de facteurs est nul quand l'un des deux facteurs est nul»
3. P répète «Un produit de facteurs est nul quand l'un des deux facteurs est nul «Or ici on a un produit de facteurs et graphiquement où est ce qu'on repère que le facteur est nul?
4. J Ben quand il est à l'origine
5. P l'origine c'est-à-dire ?
6. $\mathrm{J} O$
7. $\boldsymbol{P} O$ ah bon alors qu'est ce que tu vois en $O, O$ est ici Qu'est ce que tu vois ici en $O$ ?
8. $\mathbf{J}$ ben il y aurait un point qui serait égal à 0
9. $\boldsymbol{P}$ ah bon ?
10. J quoique
11. P quoique ... Alors qui peut nous aider, qui peut expliquer comment ce que vous récitiez en $3^{\text {ème }}$, comment on pourrait l'utiliser ici? Qui peut expliquer ça simplement? C'est-à-dire que la forme i c'est une forme qui est comment par rapport à la fonction?
12. Un élève : factorisée
13. P oui factorisée c'est a fois (x-u) facteur ( $x-v$ ). // On voudrait savoir quand est ce que c'est égal à 0
14. Un élève : quand $a=0$
15. $\boldsymbol{P}$ est ce que a peut être égal à 0 dans l'énoncé ?
16. J (et d'autres) non
17. Un autre élève : quand $x-u=0$ ou quand $x-v=0$
18. P voilà! Donc on peut dire que $a(x-u)(x-v)=0$ est équivalent à dire que $x-u=0$ ou $x-v=0$
19. $\mathbf{P}$ donc ça nous donne quoi $x=u$ ou $x=v$ et graphiquement où est ce qu'on lit $x=u$ ou $x=v$ ?
20. J sur l'axe des abscisses
21. P sur l'axe des abscisses, à quel endroit? Quels points précis? Quand la courbe...
22. $\mathbf{J}$ intercepte
23. P intercepte ...
24. J l'axe des abscisses

DIDIREM Analysis: In this episode, what is at stake is the graphical interpretation of the parameters $u$ and $v$ of a quadratic function written in factorized form, a quite difficult tasks for the students. For that purpose, and trying to overcome the difficulties, the teacher creates a trajectory going from factorization to the solving of equations, then from the solving of a factorized equation to the solving of the equations associated with its factors, and finally from this resolution to the graphical interpretation of the solution in terms of coordinates of the intersections of the graphical representation of the function and the x -axis. This is a multi-step and rather complex reasoning. Again here, this complexity has not been anticipated by the scenario and then the teacher has to improvise.

He first makes use of the didactic memory of the classroom. The association factorization-equation is indeed produced through the recall of a discursive routine used in grade 9 . Surprisingly, it works! But the mathematical signification of an association produced in such a way is problematic. All along the excerpt, the teacher is not in situation of giving responsibility of the progression of knowledge to the students. When a student gives an answer other than that expected, he modifies the question or asks other students to contribute. An adequate scenario should have taken into account the complexity of the relationship between the manipulation of the parameters and the
graph behaviour, the necessity of a reflection upon the algebraic form of the expression and its links to the graph by way of the notion of zero, and should have planned adequate period of collective reflection.

UNISI Analysis: Within this phase of the dialogue, there is not any reference to the artefact and its use. There is only a reference to the task: "comment on pourrait l'utiliser ici?" Parameters u and v are linked to transformations of the graph of the function in a much more complex way, as compared with the previous excerpt. It seems then that for the teacher, the activity with the artefact is not able to support adequately the development of an interpretation of the parameters $u$ and $v$. In addition, even if the teacher asks for students' contributions, he is not caring whether students are conscious of the discourse they are building: objective, rationale,...as if the production of the mathematical interpretation aimed at, whatever be the way it is produced, could be considered as the sign of the existence of shared, de-contextualized mathematical meanings.

## III.6. Potential offered for the theoretical landscape

## Task

The idea of task is central in both the approaches: according to the TDS, learning is a process of adaptation to a milieu organised so as to cause a disequilibrium, while according to the TSM learning is a process of construction of meanings. But both the processes are leaded by tasks. Still more, tasks are purposefully designed for leading the learning process. And there is an apparent effort for having knowledge develop through the solving of significant problems.

In coherence with TDS, a task is for DIDIREM associated with a situation (in fact a mathematical situation). The a priori analysis of its learning potential is tightly linked to that of its a-didactic potential, intended as the possibility for the situation to make pupils generate personal meanings with respect to the knowledge at stake in the situation, without the direct intervention of the teacher.

Tasks are seen and treated under different perspectives by the two teams DIDIREM and UNISI. Differences and complementarities can exist between (1) the kind of a-priori analysis of the tasks (2) the way tasks are used, (3) and maybe also the ways tasks are intended. For illustrating these it is interesting to refer to the following task used in the DIDIREM teaching experiment (beginning of the third session). We analysed above (Example 5) the dialogue between two students (and between those students and the teacher) doing this task.

The complete activity encompasses 3 tasks:
1.

2. "Indiquer la ou les méthodes utilisées pour déterminer les valeurs des paramètres";
3. "Donner une interprétation quand cela est possible des paramètres"

Pragmatically, in the analysis of the situation at stake, the focus appears to be on point 1 of the task (Activité 1 see above). This part of the task corresponds to a situation of "action" according to TDS and has an evident adidactic potential. Interaction with Casyopée is source of insights and feedbacks which can allow students to progress in a autonomous way towards the solution, and students are not dependent on the teacher for knowing if the task is solved or not. Questions 2 and 3 do not present these characteristics. Some narrative is asked and then an interpretation of the parameters. What is at stake is a relationship to knowledge in terms of formulation associated with a reflection on action, but these sub-tasks cannot be attached to a situation of formulation in the TDS sense. In fact, it is neither classical in French classrooms to observe communication and formulation situations in the sense of the TDS nor to ask students to produce narratives of their research even if this practice is well documented in the IREM literature. Most often, after a situation of action, formulation is directed by the teacher during a collective phase also comprising validation and institutionalisation. Time constraints contribute to this phenomenon as if some reduction of the didactic time had to be operated after the expansion generated by the situation of action. This general tendency has evidently affected the experimentation and the analysis of it by DIDIREM researchers.

The prominent attention given to task 1 is to some extent confirmed by the way the excerpt is presented by DIDIREM. At the very beginning, the attention is explicitly given to point 1 (point 2 and 3 not being mentioned at all), the points 2 and 3 are then presented in the transcript of the
dialogue but are seemingly treated as some kind of information needed for the reader to make sense of what happens rather than because important in their own right.

UNISI thinks the task always together with the artefact. Its analysis is meant to specify the semiotic potential of the artefact with respect to the task to accomplish. With that respect all the above tasks have crucial importance although because of their nature may contribute differently to determine the semiotic potential.

Task point 1 is analysed with the aim of identifying the personal meanings that can emerge when students use the artefact for accomplishing it. Hence the kinds of analysis of the Task 1 made by DIDIREM and UNISI are compatible: both encompass an analysis of the utilization schemes of the artefact. Both are interested in the potential change that accomplishing that task may generate in students' relationship to knowledge but the two perspectives differ.

For UNISI, Task 2 and 3 also have crucial importance because of their semiotic nature. The former is a (kind of) request to go back to the task that meant that the students are asked to rethink about the actual use of the artefact for solving the task. It is a "necessary" step for recalling the personal experience developed through that activity and may be useful to focalize on the actual use of the artefact. Moreover the request of a verbal production will lead one to a first de-contextualization expressed by written signs that provide a first elaboration of meanings. The specification of those meanings is requested by the latter task.

With that respect the formulation of Task 3 is not perfectly clear. It seems related to a particular didactic contract on going in that class. In fact, what kind of interpretation of parameters is asked to students is not made clear: an interpretation in terms of the use of Casyopée for solving the task (J:"[...] on a cherché on pouvait aller de droite à gauche ou de bas en haut [...]" Episode 10), or in terms of the pertinent mathematical meanings ( $\mathbf{P}$ : "Vous pensiez quand même à des... quel type de transformation ?" Episode 10) ?

With respect to that, DIDIREM noted in its analysis: "The teacher does not appear concerned that for students the meanings attached to the term "translation" is confined to the use of Casyopée". The gap between the interpretation in term of "translation" and the gestures in Casyopée environment seems to have also been underestimated by the DIDIREM team. Above, we noted that this underestimation is visible in the design of the scenario.

## Social Interaction: Institutionalization vs Classroom Discussion

In fact, it seems that the discussion situation in the TSM encompasses the two distinct steps of the TDS theory: the "formulation step" and the "institutionalisation" step.

In both approaches the tasks-based activity is followed by a social interaction that sees the participation of the teacher; there is also a clear intent of the teacher to make students elaborate on their experience during the task and that such elaboration be finalized to make mathematical knowledge (we would say mathematical signs) emerge. But there are important differences between the kinds of social interaction designed and enacted in the two experiments.

In the UNISI discussion (consistent with the idea of "mathematical discussion", Bartolini Bussi, 1998), the link between the accomplishment of the task and the mathematics has to be built on the
texture of personal meanings emerging from the task and just recognized from outside. The teacher's role is to use this texture to make explicit the mathematical knowledge that students should learn and to help students build the link.

This makes necessary a specific didactic contract that has to be settled in the class concerning how and why using the artefact to solve the task can be interpreted mathematically and vice versa how and why mathematics can be interpreted by referring to artefact and schemes of use.

The teacher is asked to foster students to elaborate on their own personal meanings: the evolution of students' personal meanings has to be based on previous activities with the artefact and personal meanings emerged. No classroom discussion (in the TMS sense) would be possible without the active engagement of the students, for instance for making explicit the meanings they personally developed, sharing them with classmates and the teacher. This semiotic process that is going to occur during a collective mathematical discussion involving the teacher and the students, the artefact and the mathematics, is the core of the process of Semiotic Mediation assumed by the TSM. The crucial point is how to relate shared meaning with mathematical meanings that for their nature are not negotiable but culturally established. In this respect the role of the teacher becomes fundamental as well delicate. The system of shared responsibilities that must be settled in the class community in order to make this semiotic mediation process take place can be described in terms of didactic contract as well in terms of evolution (change of) milieu. In fact, a specific didactic contract becomes indispensable in order to make the semiotic mediation process function.

## A didactic contract for the classroom discussion

The notion of didactic contract seems to be an adequate tool to study a collective discussion. In fact, in the UNISI discussion, the link between the accomplishment of the task and the mathematics has to be built on the texture of personal meanings emerging from the task and just recognized from outside. This claims the need of a specific didactic contract that has to be settled in the class concerning how and why using the artefact to solve the task can be interpreted mathematically and vice versa how and why mathematics can be interpreted by artefact tools and schema of utilization. Of course, this contract should be different from the usual contract, and some (usually considered) negative effects linked to the contract (such as the Topaze effect) may be seen under a different light.

In the analysis of another teaching experiment, carried out within the theoretical framework of Semiotic Mediation, Falcade (2006) described some rules which could be identified as part (constitutive) of the didactic contract related to Mathematical Discussion and specific of one of the classrooms involved in the experimental part of her study. Some of these rules are (seem to be) specific of the teacher style of intervention and pertinent to the specific artefact used, others on the contrary seem to be related to TSM general hypotheses and can be applicable to describe the specificity of the didactic contract in place during the collective discussions of our experimentation.

Here in the following the rules, as they are labelled and expressed in (Falcade, 2006, pp. 285-290).

- R1 : « L'activité dans l'artefact a une signification dans le monde des mathématiques, [...] une signification qu'il faut rechercher ensemble ».
- $\quad$ R2 : «tout ce qui a été fait dans l'artefact «Cabri» et tout ce qui a été dit lors de la discussion collective doit trouver une signification intégrant les différents éléments en jeu».
- R7: «Le travail de construction collective des signifiés nouveaux s'opère, tout d'abord, à partir des signifiés personnels déjà construits. Le premier souci est avant tout, d'essayer de les expliciter»
- R11: [Ce même type de contrat veut aussi que] «ce qui est posé comme «signe» reste «signe» et, en particulier, que les objets théoriques ne soient pas soumis aux mêmes types de nécessités auxquelles les objets concrets de l'artefact obéissent forcément »
- R12: «si l'artefact est un bon médiateur, l'on doit pouvoir fournir une interprétation mathématiquement consistante de ses «limites»».

In that classroom, previous teaching interventions inspired by the TSM had been enacted. Thus it can be assumed that the rules described were well established in the classroom and assured the "good" functioning of the discussion in the sense of the TSM. As said by Falcade, taking the perspective of the TDS and specifically the use of the notion of didactic contract allowed to describe the good functioning of the semiotic mediation process as it developed in collective discussions.
> "Un autre apport fondamental de TDS à la TMS a été la mise en évidence du contrat didactique spécifique, nécessaire au bon fonctionnement du processus de médiation sémiotique. A partir de la notion théorique de contrat, issue de la TDS, nous avons pu identifier les règles principales de fonctionnement sous-jacentes à ce processus. Nous avons vu, par exemple, que certaines règles, comme la règle fondamentale qui veut que « tout ce qui se passe dans Cabri ait une correspondance dans les mathématiques » (R1), ou comme la règle qui exige que «ce qui est posé comme «signe » reste « signe» et, en particulier, que les objets théoriques ne soient pas soumis aux mêmes types de nécessités auxquelles les objets concrets de l'artefact obéissent forcément» (R11) ou encore comme la règle qui postule que «si l'artefact est un bon médiateur, l'on doit pouvoir fournir une interprétation mathématiquement consistante de ses «limites» (R12), sont nécessaires à garantir que le jeu interprétatif soit aussi à la portée des élèves, c'est-à-dire toute la dévolution possible. » (Falcade, 2006, p. 386)

The situation in the classrooms which took part to the ReMath experimentation is not comparable with that of the experimental class of Falcade's study. First of all, the teacher involved in ReMath experimentation, although belonging to the same research group, was not the same of that of the previous study. Moreover, for the class involved in ReMath this was the first experience in collective discussions so that it was expected that these rules were not yet established. Nevertheless some elements can be detected recognizable as characteristic of the specific contract and in some extent can explain the success of the experiment.

## Effects related to the Didactic Contract

Within the specific didactic contract ruling the classroom discussion, some (usually considered) negative effects linked to the contract may be seen under a different light. We are referring in particular to the Topaze effect.

[^3]As usually intended, the Topaze effect consists in the progressive impoverishment of a task made by the teacher who wants to help her students to give the desired answer. This action possibly results in the student giving the "correct" answer, but to a question which is radically different from the original one which remain unanswered. The teacher reduces step by step the richness of sense of the original request, and the students tries her best to guess the desired answer so as to please the teacher. The sense of the student's answer may be no longer related to the sense of the original question. The distance between the original task and the richness of senses related to it and the actual "answered" task and the poverty of sense related to it is not perceived by the teacher and the student, who have the illusion that...

Within the context of a classroom discussion, according to the specific didactic contract of these discussions, one can found a Topaze -like effect in the semiotic tasks. But these do not function as Topaze effects. For instance, when the teacher wants to introduce a mathematical sign, or when she wants the students select specific elements of the activity with the artefact, ecc. In those cases the students may have not the possibilities of knowing what the expected answer looks like (an analysis of the characteristics of the milieu could be appropriate for that), and to some extent they have to guess what the teacher wants them say. But this time the objective is not to please the teacher per se. Neither the teacher nor the student feed the "illusion" typical of the Topaze effect: they share the objective of the intervention and according to R1 and R2 induced answers assume the sense in relation to the common goal of relating artefact use and mathematics. For this reason the given answer is not void of sense, the sense being developed, worked on, and weaved together. Never the less, such subtleties make evident the expertise required by the teacher in such collective discussions, for avoiding to fall down in the traps associated to the paradox of the didactic contract, well identified in TDS. This confirms the interest of combining an analysis in terms of didactic contract with the tools offered by TSM for understanding what is really at stake for the students in such discussions.

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## IV. MoPIX cross-case analysis

## IV.I. Identification

IOE - Institute of Education, University of London/ London Knowledge Lab
ETL - Educational Technology Lab, Education Department, PPP Faculty, University of Athens

## IV.2. Contextual elements

## IOE context

The IOE teaching experiment took place in a tertiary college in outer London. This is a large college, divided between three sites across two London boroughs, primarily providing for 16-19-year-old students but also for adult students. The provision at the site at which the teaching experiment took place is mainly 'Advanced Level' courses designed for university bound students. Because of the high-stakes nature of these publicly examined courses, there is a strong focus among both students and teachers on the demands of the examined curriculum. This affected the way in which the IOE team was able to negotiate entry into the college and the design of the teaching experiment itself.

Two of the college mathematics teachers and their divisional manager were involved in the local organisation of the experiment, discussing the mathematical focus and the nature of the activities for students. They were not prepared to allow the experiment to use time from normal scheduled lessons but encouraged students from their classes to volunteer to take part during their non-contact time. They suggested to students that they might benefit from taking part, both from mathematical preparation for the mechanics module that they were due to start studying the next term and from involvement in a novel extra-curricular activity that would 'look good' on their applications for university places. Following a presentation by one of the IOE team to the students, ten students volunteered to be involved. These students were all aged 17-18 and enrolled in the second year of an Advanced Level Mathematics course, preparing them for quantitative-based subjects at university. Unfortunately it was not possible to find a time convenient for all these volunteers so just seven of them (two girls and five boys) were able to participate in the experiment. A weekly series of eight sessions of 1 h 50 min was arranged. While all the seven students continued to participate for the duration of the experiment, not all were present each week due to other commitments such as preparation for examinations or attendance at interviews for university places.

The importance of the examined curriculum influenced the design of student activities, focusing primarily on concepts that were to be addressed in the forthcoming mechanics module. The curriculum focus of each session was made explicit to the students so that they would appreciate how their experience might relate to their future studies. In one session, a task was included taken directly from the textbook that the students would use the next term. Although the college teachers were involved in deciding the focus of the activities, they were not able to participate in the teaching due to other commitments. Teaching was conducted by a member of the IOE team, while another member of the team acted as researcher.

Each session took place in the same classroom. This room was equipped with PCs around the periphery and a data projector. The students sat around tables in the middle of the room, each equipped with a tablet PC. The data projector was occasionally used with one of the tablet PCs to demonstrate specific aspects of MoPiX to the whole group or as the focus for a whole group discussion of a problem. Most of the time, however, the students worked in pairs, sometimes each using their own tablet PC and sometimes putting one of them aside in order to work together. One of the students preferred generally to work by himself but the others discussed and collaborated well. There were peer interactions both within work pairs and between pairs. Due to absenteeism, pairings of students were not fixed, though there was a high degree of stability.

## ETL context

The ETL teaching Experiment was conducted in a Secondary Vocational Education School in Athens. The Vocational Education schools in Greece comprise three grades and accept students who have just graduated from the Lower Secondary Education schools ( 15 years old) as well as older students ( 16 to 19 years old or even adults) who wish to acquire professional competencies in a certain domain such as mechanical or electrical engineering. At the end of the $3^{\text {rd }}$ year the students are granted a vocational certificate and may enter the labour market directly or sit special exams for entering Tertiary Education and continue their studies in a department relevant to the orientation of their certificate. The eight $3^{\text {rd }}$ grade students we wished to work with were keeping their options open but since it was still early in the year, little was at stake for them at that time. Even so, as both alternatives were directly linked to their domain of expertise (mechanical engineering), they all shared an increased interest in engineering activities such as modelling and in novel computational environments designed for this cause. Thus, when an ETL member asked them if they would like to engage in a project that would involve modelling activities using a specially designed computational environment, they all voluntarily agreed to participate.

The curriculum in Vocational Education schools encompasses specialization subjects relevant to the orientation the students have selected in the first year of their studies as well as general knowledge subjects. Modelling activities are very common in mechanical engineering education and they were more or less explicitly included in many specialisation subjects the students were taking during their three-year tuition without, however, having a strong connection with the use of technology. Mathematics, on the other hand, is also included in the curriculum (as a general subject instructed in classroom) but it is mainly perceived as a separate body of knowledge the students must acquire to be admitted in Higher Technological Education Institutes. Recognising in this context a gap between mathematics and several engineering subjects requiring the use of mathematics, we designed a set of activities that would bring mathematics inside genuine engineering procedures (such as modelling) using a specially designed computational medium. Since the design of our activities was based on the idea of using the technology so as to by-pass conventional practices such as keeping apart general and specialisation subjects and sticking to the curriculum's predefined content and proposed teaching methods- we didn't consider it necessary to directly connect the students' activities to specific curricular goals.

The regular mechanical engineering teacher to whom we suggested to participate in the design of the activities and the experiment declared that he was not interested in such a perspective but agreed to provide us his regular school hours ( 45 minutes each) for the experimentation process,
recognizing a mapping between the proposed activities and engineering education. Hence, the local organization of the experiment was held by two ETL members, one of which had been a teacher in the implementation school for more than seven years. This fact allowed us to have full access to the premises and equipment and facilitated our communication with the school administration, the computer lab director and the rest of the school teachers who all agreed to yield us the computer lab for as many school hours as requested. Because of her previous experience as a mechanical engineering teacher in the school, during the experiment, this ETL member acted as a teacherresearcher, while the second one as a co-researcher.

The computer lab in which the experimentation process took place was equipped with PCs around the periphery -all having access to the Internet- and a data projector. The eight 3rd grade students (17 years old), studying mechanical engineering formed groups of two or three and worked together for 25 school hours ( 10 sessions). Each workgroup members shared a PC, a MoPiX manual in Greek and a notebook for recording their ideas and remarks. Apart from working in groups, at certain phases of the experimentation, the students gathered around a large table at the centre of the room and participated in plenary sessions having at their disposal a PC and the data projector. The plenary sessions allowed them to discuss and share ideas with other workgroup members, exchange artefacts and/or work collaboratively for a common goal.

Designing activities that would foster communication within the workgroups and among the workgroups aimed at providing students the opportunity to discuss, argue and negotiate with their peers and engage in joint decision and meaning-making processes at different social levels within their classroom. This kind of orchestration is not considered to be a regularity in the Greek educational system, since the respective schooling paradigm doesn't encourage the establishment of social norms that support collaboration, inquiry and argumentation.

Table 1: Comparison of contextual elements

IoE/LKL
ETL

Tertiary College (University bound)
Secondary Vocational Education

7 students, studying Advanced level 8 students, studying Mechanical Engineering Mathematics
explicit connection to the standard curriculum

Year 13, 17 years old

Workgroups of 2, each student with a Tablet Workgroups of 2 or 3, sharing a PC PC
not directly connected to standard curriculum

12th grade, 17 years old

Wokgoups of 2 or 3 , sharing a PC

8 sessions - 15 school hours

Interactions within and between workgroups

2 IOE researchers, one also acting as teacher

10 sessions - 25 school hours

Interactions within and between workgroups

2 ETL researchers, one also acting as teacher

## IV.3. Theoretical frames

## IOE theoretical frame

The primary theoretical framework adopted by the IOE team involved in the teaching experiment is multimodal social semiotics (Kress, Jewitt, Ogborn, \& Tsatsarelis, 2001; Kress \& van Leeuwen, 2001; O'Halloran, 2005). This informed the design of MoPiX, the design of the teaching experiment and the analysis of its results. Although originating in linguistics, this theoretical framework challenges the primacy of language as a means of communication and meaning making, highlighting the different potentials for meaning offered by different modes of communication. MoPiX is designed to provide a multi-semiotic environment involving formal notation of equations and visual animated models. The trace of motion of an animated object can also be perceived as a Cartesian graph. In addition, the pedagogical plan and the organisation of the teaching experiment were designed to enable students to communicate using pencil and paper-based representations involving conventional or informal notations or diagrams, using 'natural' language in face-to-face speech and by sharing MoPiX objects, equations and models electronically. Each of these modes of communication involves its own distinctive system of elements, grammar and meaning potential. Moreover, interaction between such different modes creates further opportunities for meaning making. In designing the pedagogical plan we thus saw the multi-semiotic nature of the environment as providing rich possibilities for students to interpret and to express mathematical ideas.

Social semiotic analysis of communication is not concerned with the cognition underpinning the communication or with the intentions of those involved but with the meanings produced in the social interaction. Our reformulation of the common research question thus focussed on the ways in which students operated with the various modes available to them and the relationships between their semiotic activity and mathematical meanings relevant to the educational goals of the teaching experiment:

What concepts of motion are represented through students' semiotic activity in the context of use of MoPiX?
-How do students operate in MoPiX with the variables $x$ and $y, V x$ and $V y, A x$ and Ay? In order to achieve what goals?
-What forms of language and other modes do students use to communicate about velocity and acceleration as they work to construct animations and to interpret sets of equations and graphs?
-What choices do students make between and within the semiotic systems offered by MoPiX and the context of its use in order to communicate meanings related to motion?

> Do students' communications about velocity and acceleration outside of MoPiX vary through the course of their experience with MoPiX?

The data collected and the methods of analysis also reflected this perspective, drawing together student productions in different modes (paper-and-pencil, MoPiX programming, speech, gesture) and tracing both the meanings produced through each mode of communication and the interactions between modes.

In considering specifically mathematical aspects of multimodality, we refer to Duval (2006), who argues (from a different semiotic tradition) that conversion between semiotic systems (which he names representational systems or 'registers') is of fundamental importance to mathematical learning. Conversion demands that the student distinguishes what is mathematically relevant in each system and separates the mathematical object from its representation. The MoPiX environment not only demands that students engage in conversion (using different forms of representation for the 'same' mathematical object) but also that they actively use the representations available in the system of equations to effect changes in the visual forms of representation. In the opposite direction, the process of 'debugging' faulty animations again demands conversion: identifying those equations responsible for the 'buggy' behaviour. According to Duval's principle, we hypothesised that activities in this environment would enable students to develop their understandings of algebraic notation and of definitions of motion. This hypothesis, however, did not form a focus for our research as our overarching social semiotic perspective directs our attention to communication and meaning making in social contexts rather than to investigation of individual cognitive development.

Social semiotics is not a theory of learning. While it can suggest how students may make sense of the multimodal texts they experience in the classroom and hence suggests some characteristics of good learning environments, it is not sufficient by itself to inform the design of activities for learning. The design of MoPiX itself and of the pedagogical plan used in the teaching experiment were also influenced by a broadly constructionist theoretical frame. The constructionist approach to learning (Harel \& Papert, 1991; Kafai \& Resnick, 1996) promotes investigation through the design of microworld environments. MoPiX is conceived as a constructionist toolkit (Strohecker \& Slaughter, 2000), a dynamic visual environment that supports construction activities in social contexts, based on these principles. Learners use the fundamental elements of the microworld (equations and objects whose properties and behaviours are defined by the equations assigned to them) to build objects and models with new sets of properties and behaviours. They may then activate their constructions to investigate them, forming and testing hypotheses about their behaviours. However, interaction with physical representations is not by itself sufficient for effective learning, learners need to make sense of their experiences of manipulating representations in the context of social interaction with peers and with teachers in order to be able to challenge and
test alternative conceptualisations and forms of reasoning. The design of MoPiX facilitates collaboration through sharing models electronically, while the teaching experiment sought to support and encourage such social interaction through setting tasks that demanded communication between peers (both electronically and in face-to-face discussion) and through the arrangement of the physical environment.

Students' informal ways of experiencing the world often prepare them poorly for understanding scientific principles in the area of mechanics. Indeed, students' intuitive assumptions and arguments about velocity, acceleration, forces, etc. are often discontinuous with the principles and styles of reasoning underpinning the Newtonian model and everyday physical experience does not sufficiently challenge these assumptions and arguments (Eckstein \& Shemesh, 1989; Graham \& Berry, 1990). The objects and equations of the MoPiX microworld are designed to behave in mathematically coherent ways. This provides an environment that, by exploring and building models within the microworld, allows students to construct mathematical meanings. By setting students tasks to build dynamic models that behave in ways consistent with their experience of the physical world, students' intuitions about motion, forces, etc. may be challenged.

## ETL theoretical frame

Bearing in mind the contextual elements described in the previous section, the ETL team adopted in the design of its teaching experiment a constructionist theoretical perspective (Harel \& Papert, 1991; Kafai \& Resnick, 1996). Even though constructionism has been perceived as an individualistic theory of learning, the ETL team studied students' communication and argumentation in small group collaborative activity. At a first glance, the constructionist perspective appears to be strongly connected to vocational education, in the sense that knowledge construction in vocational education is considered to be accomplished through activities that engage students in the hands-on construction of external -mainly tangible- artefacts, such pulleys and gears. However, the hands-on activities proposed by this paradigm remain fragmented as students rarely have the chance to immerse in genuine designing and modelling activities and build artefacts that are personally meaningful to them. The products they are asked to construct are predefined and uniform while the procedure to follow is predetermined. This fragmentation is also advocated by the fact that formal mathematics is taught in school as a separate body of knowledge, detached from the students' vocational activity. The vocational mathematics, on the other hand, embedded in the students' routine activities, remain invisible as students usually apply mathematical procedures (such as calculating values using an algorithm or reading a graph) rather mechanically, without conceptualising the mathematical models underpinning their actions. The various professional ICT tools used in these cases only sustain this fragmentation. Using the constructionist perspective as a vehicle to infuse innovation within the existing vocational education schooling paradigm and address the fragmentation of the notion of "knowledge construction" in this context, ETL designed activities that use MoPiX, a specially designed computational medium, as means to bring visible mathematics at the core of genuine engineering activities.

MoPiX is mainly perceived by ETL as a learning environment designed to allow students to engage in constructionist activities. The learners in MoPiX have the opportunity to explore, manipulate and build animated models representing different phenomena and situations, engaging in the way in meaning making processes. However, as the models' behaviours are defined and controlled by
algebraic equations, the real driving force behind any constructionist activity in MoPiX is the use of mathematical formalism. Students employ equations as means to convey meaning while exploring, manipulating, designing and developing virtual models. As the equations attributed to the objects do not constitute "black boxes", unavailable for inspection or modifications by the user (for a discussion on black and white box approaches see Kynigos 2004), the learner in MoPiX has deep structural access (diSessa 2000) to the mathematical models underpinning the behaviours animated on the screen. Thus, ETL considers MoPiX as an environment that brings mathematical formalism to the foreground in a learning situation where formalism is put to use in a meaningful way. Against a rationale supporting that with the help of digital technologies we can at last by-pass the need to use formalism in order to access mathematical ideas the ETL team also addresses students' problem of perceiving formalism as meaningless (Dubinsky, 2000) but in a different way: to find uses of technology to change the role of formalism so that students can put it in use (and the mathematics it embeds) to make constructions and models. In this way, instead of considering students' use of mathematical formalism as distinct or even an obstacle to meaning generation, these two can be perceived as interwoven (Kynigos and Psycharis, 2003). The ETL team thus focused on the ways in which the MoPiX formalism was used by the students studying both what they did with it and their verbal exchanges as they were collaborating to build their models.

What has also been interesting for the ETL team is the potential to design in the MoPiX computational environment half-baked microworlds (Kynigos 2007) i.e. microworlds that incorporate an interesting idea but are incomplete by design so as to invite students to deconstruct them, build on their parts, customize and change them, eventually constructing a new artefact that could be distinctly different than the original one. Half-baked microworlds are by nature designed for instrumentalisation (Guin and Trouche 2002) since they serve as intriguing starting points and idea generators, stimulating students to transform them from a plain artefact (i.e. a piece of software constructed by humans) to an instrument (Rabardel 2001), engaging in the way in constructionist meaning-making processes. However, since MoPiX fosters collaboration among peers through model sharing and exchanging, we particularly focused on the idea of developing half-baked microworlds for which the instrumentalisation process will take place in a collaborative context.

Taking into account those aspects of the MoPiX computational environment and drawing on the socio-constructivist paradigm, ETL designed a Pedagogical Plan that provided students the opportunity to interpret, manipulate and use the available mathematical formalism in the process of creating and controlling animated models while collaborating with their peers at different social levels. The "Juggler" half-baked microworld designed for the Pedagogical Plan invited students to deconstruct it so as to explore its functionalities and (re)construct it to embed their own ideas, engaging in the way in meaning making processes. The meanings generated, however, were not predefined by the researchers so as to a priori correlate them with specific curricular goals, but were considered to be emergent and shaped by the students' mathematical and social activity. Thus, the ETL PP addressed open didactical goals that did not directly correspond to specific National Curriculum educational goals.

## IV. 4 Comparison of didactical functionalities

Differences in the educational goals of the two teaching experiments and in their theoretical orientations led the two research teams to focus on different sets of didactical functionalities. These
are summarised in Table 2. The one aspect that was clearly significant for both teams was related to use of the MoPiX library of equations. Not only was this characteristic of the DDA salient for both teams but it was also perceived to serve a similar function for both in relation to students making sense of the equations in order to achieve the respective educational goals. This function was facilitated in both cases by the forms of pedagogy used, shaped by the shared constructionist framework, that enabled students to form their own approaches to problems and, indeed, to define their own problems.

The other didactical functionalities highlighted by the IOE teaching experiment focus on the specific forms of representation provided by MoPiX, the epistemological consequences of these representations and the semiotic activity of students in relation to these representations and to other social and semiotic aspects of their environment. It is apparent that these sets of didactic functionalities become salient to the IOE team through the lens of their social semiotic framework.

## Table 2: Didactical Functionalities

| Team | characteristics of the DDA | educational goals | modalities of use |
| :---: | :---: | :---: | :---: |
| Both IOE and ETL | The library of equations provided a basic set of equations with very limited variation. | The library allowed students to get acquaint with the MoPiX environment, interpret and use equations that had an easily identifiable pattern in the X and Y axis and were classified in categories whose names revealed more or less the assigned behaviour. <br> This allowed students to develop a strategy of paying attention to the meaning of a limited sub-set of equations while using others in a 'black box' mode | The limited variation in the equations challenged students not merely to confine themselves in using specific ready-made equations but to seek for ways to define new ones in order to express ideas that were not accurately described by those provided in the library. <br> This strategy was enabled in both teaching experiments by pedagogies that encouraged students to develop their own problem solving approaches rather than directing them to use specific methods. |


| IOE | The symbolic representation of motion is always separated into horizontal and vertical components. | This form of representation provided students with an alternative language that moved away from the 'everyday', enabling new analytic and quantitative approaches to defining and describing motion. | The language of components was used by teachers and students throughout the teaching experiment both in formal 'MoPiX language' (e.g. 'Vx') and in less formal adaptations (e.g. 'x velocity'). |
| :---: | :---: | :---: | :---: |
| IOE | The graphical  <br> representations of  <br> MoPiX support  <br> interpretation of  <br> graphs as static <br> patterns or as traces of  <br> motion, while the <br> symbolic   <br> representations   <br> (always expressed in  <br> terms of a parameter   <br> of time) support the <br> dynamic   <br> interpretation.   | The dynamic interpretation supported by both symbolic and graphical representations seems more aligned with our educational goals. | However, students' adoption of static or dynamic interpretation was affected by interaction with the other semiotic resources available in the immediate context. |
| ETL | The mathematical formalism as means of expression | Interpret, manipulate and use the mathematical formalism to convey meaning in the process of designing, creating and controlling animated models. | Students used the mathematical formalism as means of expression in the process of exploring, manipulating, designing and building animated models. |
| ETL | Deep structure access to the mathematical models underpinning the behaviours animated | Linking the animation generated on the screen to the model's symbolic facet, interpreting ready-made or new equations in terms of their structure and content. | As students attempted to debug flawed animations they engaged in back and forth processes of constructing a model predicting its behaviour, observing the animation generated, identifying the equations that are responsible for the "buggy" behaviour and specifying which and how particular parts needed to be |


|  |  |  | fixed. |
| :---: | :---: | :---: | :---: |
| ETL | Potential to develop half -baked microworlds | Deconstructing the microworld's model so as to link the algebraic equations to the animations generated. Use the available mathematical formalism to build new equations so as to reconstruct the model, attributing it new behaviours. Possibly extending the available mathematical formalism, building a new vocabulary. | Half-baked microworlds were used Designed for instrumentalisation incorporate an interesting idea but are incomplete by design. <br> They constitute intriguing stating points and idea generators. Students deconstruct it, customized and changed it, eventually constructing a new artefact that could be distinctly different than the original one. |

## IV. 5 Results of the cross-case analysis together with illustrative examples

We present two examples of episodes taken from our data, the first from the IOE teaching experiment and the second from the ETL teaching experiment. An analysis of each episode is provided from the perspective of the experimenting team and a cross-analysis from the perspective of the other team. A commentary is then presented, highlighting the similarities and differences between the two analyses.

## Episode 1: Changing the direction of motion (familiar IOE teaching experiment)

In the seventh session, students were introduced to the idea of acceleration applied to an object at an instant. They experimented with applying acceleration equations of the form $A x$ (object_1,20)=3 (applying an acceleration of 3 units in the horizontal direction when time is 20), observing the effect as a sudden change in direction. They were then posed the task of using such acceleration in order to draw a square. In an earlier session students had worked on the outwardly similar task of drawing shapes (not including a square) by making changes in velocity. It was here that Ron decided to start. Rather than using acceleration, he first used velocity equations to turn the corners of his square then started the task of drawing a square using acceleration equations..

In Table 3 and Table 4 we present an overview of Ron's solution process
Table 3: Ron's construction of a square using velocity to change direction

| time | duration | sub-problem | trials | notes |
| :--- | :--- | :--- | :--- | :--- |
| 5:00-7:00 | 2 min | adding basic equation sets for <br> vertical then horizontal movement | 4 | incomplete <br> trialled then added to; |


|  |  | to object 1 |  | final complete set not trialled |
| :---: | :---: | :---: | :---: | :---: |
| 7:00-10:30 | 3 min 30 | adding $\mathrm{Vx} / \mathrm{Vy}$ for $\mathrm{t}=20$ <br> and Vy for $\mathrm{t}=40$ | 4 | strategies: <br> change sign <br> change size <br> correct solution not achieved |
| 11:12-12:12 | 1 min | editing $\mathrm{V} \times / \mathrm{Vy}$ for $\mathrm{t}=20$ | 1 | correct |
| 12:12-12:50 | Omin38 | adding $V \times$ for $\mathrm{t}=40$ | 1 | correct |
| 12:50-13:40 | Omin50 | adding $\mathrm{V} \times / \mathrm{Vy}$ for $\mathrm{t}=60$ | 1 | correct |
| 13:40-14:25 | Omin45 | adding $\mathrm{V} \times / \mathrm{Vy}$ for $\mathrm{t}=80$ | 1 | correct |
| 14:25 | general statement explaining to his partner how to use $V x / V y$ to turn $90^{\circ}$ - change from (30) to (0-3) <br> " When you want it to turn you got to say at 20, or whatever you want, Vy equals zero, $V x$ equals three, whatever what happened, at $40, \mathrm{Vx}$ equals zero, Vy equals minus three, at ... " |  |  |  |

Table 4:Ron's construction of a square using acceleration to change direction

| time | duration | sub-problem | trials | notes |
| :--- | :--- | :--- | :--- | :--- |
| $22: 48-23: 00$ | Omin12 | adding basic equation sets for <br> vertical and horizontal motion to <br> object 3 | 1 | correct |
| 23:00-34:00 | 11 min | adding Ax/Ay for $\mathrm{t}=20$ | 14 | strategies: <br> change sign of Ax or Ay <br> change size of Ax or Ay <br> (double; very large; very <br> small) <br> remove Ax or Ay |
| $34: 14-36: 33$ | 2 min19 | adding Ax/Ay for $\mathrm{t}=40$ | 3 | Ax added first then trialed <br> Ay added and trialed <br> Corrected Ay (change sign) <br> added and trialled |
| $36: 33-38: 00$ | 0min27 | adding Ax/Ay for $\mathrm{t}=60$ | 2 | Ax/Ay both added then <br> trialled <br> Corrected Ay (change sign) <br> added and trialled |
| $38: 00-40: 00$ | 2 min | adding Ax/Ay for $\mathrm{t}=80$ | 5 | Ax/Ay added then trialled <br> corrected Ay (change sign) <br> added and trialled <br> corrected Ax and Ay <br> (change size of both) <br> added and trialled |
| $40: 00-44: 16$ | $4 \min 16$ | editing Ax/Ay for $\mathrm{t}=80$ - informed <br> by inspecting equations | 2 | object 1 flipped and <br> inspected <br> corrected Ax (change sign) |

$\left.\begin{array}{|l|l|l|l|l|}\hline & & & \begin{array}{l}\text { added and trialled } \\ \text { pause (30sec) } \\ \text { several } \\ \text { removed to stage } \\ \text { pause (60sec) }\end{array} \\ \text { corrected Ay (change size) } \\ \text { added and trialled }\end{array}\right]$

Table 5 presents a comparison of Ron's processes as he attempted the two tasks.
Table 5: Summary of Ron's solution processes

| Velocity | Acceleration |
| :--- | :--- |
| tentative start, adding and trialling subsets of <br> basic motion equations for first side of square | confident start: basic equation set added <br> immediately |
| trial and improvement (4 trials) to achieve first <br> turn; change sign/ swap values strategy | trial and improvement (14 trials) to achieve first <br> turn - wide range of strategies |
| rapid, accurate addition of equations for <br> subsequent turns without trialling | subsequent turns: $x$ and y components added and <br> trialled separately; only sign corrections needed |
| general statement for producing right turns | inspection of equations and extensive pause for <br> thought |

The original analysis by the IOE team and the cross-analysis by the ETL team are presented side-by-side below.

## Analysis and Cross-analysis of Episode 1

| IOE analysis | ETL cross-analysis |
| :---: | :---: |
| In the seventh session, students were introduced to the idea of acceleration applied to an object at an instant. They experimented with applying acceleration equations of the form $A x\left(o b j e c t \_1,20\right)=3$ (applying an acceleration of 3 units in the horizontal direction when time is 20 ), observing the effect as a sudden change in direction. They were then posed the task of using such acceleration in order to draw a square. In an earlier session students had worked on the outwardly similar task of drawing shapes (not including a square) by making changes in velocity. It was here that Ron decided to start. Rather than using acceleration, he first used velocity equations to turn the corners of his square. | Initially Ron attributed to his object ready-made equations that he found in the "Equations Library" classified under the "Horizontal" and "Vertical Motion Equations" categories. As he had already gained familiarity with the equations of those two categories and the meaning their symbols conveyed, Ron carefully selected only the equations that would assist him in drawing a shape and disregarded other ones he found classified in the same categories (e.g. the "amIHittingtheGround" equation). Having worked before with shape drawing using motion equations, Ron chose to bring in this task a strategy that he had previously followed and had been proven to be successful. Thus, although he initially attributed to his object an acceleration equation, he preferred to investigate the role of the velocity equations -instead of acceleration equations- in drawing a square, which seemed to be consistent to what he had achieved up to that point. |
| After some initial hesitation he created his object, assigned it a basic set of motion equations and, after a short period of trial and improvement using strategies such as changing the signs or swapping the values of $V x$ and/or $V y$, found the necessary equations to turn the first corner of the square. He then completed the other corners of this square efficiently and accurately. | Having identified the meaning the symbols in the velocity equations conveyed and having particularly articulated an understanding regarding the variable of time and its specific role in the equations, Ron performed a series of changes not only on the right part of the equation substituting one numerical value to another, but also on the left part of the equation substituting the variable of time to specific arithmetic values. The continuous changes in the values as part of his experimentations with the velocity equations did not just confine in substituting one arithmetic value for another but also |


|  | involved sign changes to signify changes in the object's direction. |
| :---: | :---: |
| Ron's initial systematic trial and improvement strategy of changing the sign or swapping the values of the new velocity worked well in this case because of the nature of the relationship between horizontal and vertical components of velocities of perpendicular motions. | Ron's initial systematic trial and improvement strategy of changing the sign or swapping the values of the new velocity worked well also because he had deep structure access to the symbolic facet of the model animated on the screen. In the process of debugging his model, Ron pressed the "Play" button to observe the animation generated and flipped his object to identify the equation responsible for the buggy behaviour several times, developing in the way meaningful connections between the mathematical formalism and the graphical/visual representation of the model. Specifying each time which equations needing to be fixed, Ron performed a series of changes editing the symbols of the velocity equations. |
| On completing the task, his growing confidence was apparent as he explained spontaneously to his partner how to make an object turn right-angled corners. | Mentioning to his partner that he could use " 20 or what ever you want" as the time point at which changes to the values of velocities should be made so as to bring changes the object's direction, Ron seems to have reached a higher level of abstraction as he appears to have identified " 20 " not as a fixed arithmetic value necessary in drawing any square but as a value that could be of the user's choice. |
| He then started the task of drawing a square using acceleration equations. This task was clearly seen as parallel to the one he had just completed as he kept this model of a square formed by using changes in velocity on the screen and constructed his second model next to it, running both simultaneously and comparing the results at each stage. After making a more confident start to creating the basic motion of the new object, Ron then ran into difficulties. | Ron used the model he had developed before as a starting point to go further with his experimentations with the acceleration equations. Thus, he initially attributed to his new object the equations he used to make the first object move upwards and draw one of the square's sides bringing in once again strategies that he had successfully employed before. |

As he tried to turn his first corner, the change sign/swap values strategy no longer worked. At first he did not appear to see how to overcome this, resorting to alternative strategies such as doubling and trying extreme large and small values of acceleration. These strategies focused only on the values of the acceleration and his exploratory attempts appear to take no account of the desired values of the velocity. After 11 minutes and 14 trials he succeeded in finding the values of acceleration needed to turn the first corner. Having achieved this, he proceeded to turn the other corners successfully and relatively efficiently, having to make only minor corrections.

As he had already created a model in which he used the velocity as the varying quantity inducing changes to the object's direction, Ron focused on producing a new model having the exact same effect to the object's direction, using this time the acceleration as the varying quantity instead of the velocity. The strategies he selected to use in this case also seemed to differentiate. In order to produce the first turn, Ron made several changes to the arithmetic value on the right side of the $X$ and $Y$ acceleration equations. Nevertheless, these changes seemed to be coherent as he moved from doubling the value of the acceleration he had previously attributed to his object to giving extremely large and small values, observing in each case the animation generated. At that point Ron didn't seem to have developed concrete links between the changes in acceleration and the changes in direction. The fact however that any actions he performed to the model's symbolic facet (e.g. editing/modifying or at several times inserting/removing acceleration equations) produced a direct change to the visual result generated on the Stage, gave Ron the opportunity to gradually move to a more solid conceptualization of the mapping between direction and acceleration and to continue his construction having identified a pattern of changes to be made so as to make the object turn.

Coming to the final corner, Ron seemed to have realised that the pattern he had previously indentified and successfully used wouldn't make his object come to a stop. Thus, instead of making any attempt to attribute the " 3 " or " -3 " value to the $X$ and $Y$ axis acceleration as he did before, he decided to explore the potential of attributing to both accelerations the " -6 " value. At this point, Ron seemed not to
each case, apparently comparing the values of velocity and of acceleration at each of the corners. With significant pauses for thought, he succeeded in adding correct acceleration equations without further trials. Finally, having completed a correct model, he spent time inspecting the equations of the original model built using changes in velocity, pointing to the various values of velocity as if calculating what acceleration would be needed to achieve the same effect.

Ron's earlier experience with MoPiX enabled efficient association of change of direction of motion with change in values of horizontal and vertical components of velocity. However, his initial use of acceleration to achieve a similar effect did not appear to make use of the concept of acceleration as change in velocity. Engagement with the symbolic mode in MoPiX and interaction between this and the animation mode enabled him to complete the task successfully. His final period of inspection of the sets of equations for both objects, pointing in turn to the velocity equations used at each corner of the original model, suggests a move towards a focus on acceleration as change in velocity.
have exactly specified the way in which changes in the acceleration equations affected the velocity of the object (so as to make the necessary changes in the acceleration values and cause the object stop at a specific time point). In this case the deep structure access the user has in the MoPiX environment was again proven to be handy as Ron flipped both his objects to inspect the symbolic facet of the two models. Recognizing an equivalence between these two models, Ron seemed to be calculating at his second model the values of the $X$ and $Y$ velocities for each time instance (through the " $V x(M E, t)=V x(M E, t-1)+A x(M E, t) " \quad$ and the $\quad V y(M E, t)=V y(M E, t-$ 1) $+\mathrm{Ay}(\mathrm{ME}, \mathrm{t})$ " equations) and compare them to the ones that appeared on his first model in the form of " $V=$ an arithmetic value".

Initially Ron seemed to be attempting to make connections between a varying quantity (the velocity in the first case and the acceleration in the second) and the changes in direction to be produced so as to make his object turn. He started inserting and changing arithmetic values on the velocity and acceleration equations and at each time started the animation to observe on the screen the graphical effect of his actions. The deep structure access and the linked representations gave Ron the opportunity to develop an understanding between the changes in direction (the visual effect) and the modifications made on the acceleration and velocity equations (the manipulations performed using the mathematical formalism).

However, at the last part of his experimentations with the acceleration equations, Ron seemed to develop an understanding

|  | regarding the relationship between the two varying quantities (i.e the velocity and the acceleration) in drawing the squares. Up to that point, he didn't seem to have made any connections between velocity and acceleration as in order to construct the first model he merely manipulated and modified velocity equations, while in order to construct the second one, he solely used acceleration equations. Any modifications made to each one of them were regarded in isolation. Flipping the two objects, the symbolic facets of the two models were put the one next to the other. Needing to calculate the velocity at each time instance at the second model and match the values calculated to the ones that appeared at the first one, Ron came to use the " $V x(M E, t)=V x(M E, t-1)+A x(M E, t) "$ and the " $\mathrm{Vy}(\mathrm{ME}, \mathrm{t})=\mathrm{Vy}(\mathrm{ME}, \mathrm{t}-1)+\mathrm{Ay}(\mathrm{ME}, \mathrm{t})$ " equations which describe the relation between the acceleration and the velocity at each time instance. |
| :---: | :---: |

## Commentary from ETL perspective on the two analyses of Episode 1

Analysing the data of the "Changing the direction of motion" episode, particular attention was paid by ETL to the ways in which the student used the available mathematical formalism in order to construct and manipulate animated models in the MoPiX environment. ETL has put emphasis on the student's experimentation process with the MoPiX equations, while the final production of the task (i.e. the square drawn using motion equations) as well as the curricular goals set in the task design (i.e. the generation of meanings about acceleration as a change in velocity) were not considered to be of primary importance for the analysis of the episode. Moreover, although it is apparent from the IOE analysis that the student engaging in the task did generate meanings about the notion of acceleration as a change in velocity, ETL chose not to focus on the generation of meanings itself -as one of the students' achievements- or the ways they were generated through the student's interaction with different semiotic systems, but primarily on the ways the mathematical formalism served in this process as a resource for the students.

Drawing on our own experiment and analysis -where the students constructed meanings for the role of equations in the models animated by employing the available mathematical formalism in specific ways- we tried to indentify a mapping between the IoE student's and the ETL students' strategies and approaches in using and manipulating MoPiX equations to make sense of the situation at hand. Thus, going through the IOE data, we focused on particular incidents in which the student seemed to interpret MoPiX equations as well on incidents in which he performed changes to ready-made equations he found in the "Equations Library" or to equations he constructed himself (substituting for example arithmetic values for other ones and/or substituting the variable of time for specific arithmetic values). The meaning generation process in both the IoE and ETL experiment is perceived by ETL as being in close relation to the equation interpreting and editing procedures in which the students engaged as they attempted to define and control the behaviours of their models.

However, in the analysis of the IoE episode, ETL also gave emphasis to specific characteristics of the DDA as having an important role in student's meaning making processes. The deep structure access MoPiX allows his users to have to the models animated and the linked representations (i.e. the symbolic and the graphical) were considered to be two of those characteristics. In certain phases of the experimentation process we identified incidents in which the IoE student, after starting and observing the animation, flipped his object to look for the equations that needed to be fixed so as to produce the desired visual effect, developing in the way meaningful connections between the mathematical formalism and the graphical/visual representation of the model. In other cases, flipping objects and putting side by side the symbolic facets of two equivalent models, allowed the IoE student to inspect and compare the equations comprising these models, using in this way the mathematical formalism as means to develop an understanding for the relationship between the specific varying quantities appearing in both models. In both the IoE and ETL experiment, the deep structural access and the linked representations are perceived by ETL as two of the DDA's characteristics that support students in their explorations and experimentations with the mathematical formalism and have a substantial role in the students' meaning generation processes.

On the other hand, the IoE analysis of the "Changing the direction of motion" episode doesn't clearly refer to specific characteristics of the DDA as playing a crucial role in the students' meaning
generation process. Nevertheless, it is apparent that the fact that the symbolic representation of motion in MoPiX is always separated into horizontal and vertical components was taken into account in the analysis as an element supporting students in associating changes to the animation generated on the screen to the changes performed by the student to the acceleration and velocity equations.

## Episode 2: "gineprasino" (ETL alien teaching experiment)

Two students working together with the "Juggler" half-baked microworld set themselves the goal of programming a moving ball to change colour as it passes another object.

## Analysis and cross-analysis of episode 2

|  | ETL data and analysis | IOE cross-analysis |
| :---: | :---: | :---: |
| 1 | S1 What I want to happen is that: when the ball is above the ellipse to become red and when it is below the ellipse to become green. I don't care about when it hits [i.e. the ground]. Can we do this? | everyday language used to describe the goal of the activity |
| 2 | S2 You have to define something. How did you define the plane which is the ground? How did you define that on the right side there is a wall and that you can't go beyond this wall? [The "ground" and the" wall" are elements of already existing equations that the students had used]. | related to the MoPiX meta-action 'define' <br> 'ground', 'wall' are boundary terms: there is a convergence between their everyday use and their use as components of MoPiX terms |
| 3 | S1 [To R1] Excuse me ... The $x$, $y$ coordinates. Can't the environment recognize them? Their values. Where the objects are situated. Can't it recognize them? | mathematical $x$, $y$ coordinates - mathematises the everyday expression 'where the objects are situated' |
| 4 | R1 Yes. |  |
| 5 | S1 It can recognize them. So I can say that I want this [i.e. the ball] to change colour. | everyday language to describe the goal |
| 6 | R1 Yes? |  |
| 7 | S1 When it is situated in a $Y$ below the $Y$ of this one for example [i.e. the ellipse]. | mathematical use of $y$, transition from the everyday expression at turn 1 |
| 8 | R1 You know ... I'm thinking ... Will the ball know when it is below or above the ellipse? | R1 introduces anthropomorphism "will the ball know" |
| 9 | S2 That's what we will define. We will define the Ys. |  |
| 10 | S1 This. The: "I am below now". How will we write this? | adopting the anthropomorphism of R1 at turn 8; this is also |


|  |  | compatible with the MoPiX term "AmIHittingGround" <br> The association between the desired goal and previous activity with 'ground' and 'wall' continues from turn 2. |
| :---: | :---: | :---: |
| 11 | S2 Using the $Y$. Using the $Y$. The $Y$. That is: when its $Y$ is 401, it is red. When the $Y$ is something less than 400, it's green! | further mathematisation of the goal, using a specific value 400 (as a generic example?), further transition turn 1 (S1) to turn 7 (S1) to turn 11 (S2) |
| 12 | S1 Let's start on that. Let's do it. |  |
|  | Starting developing the equation on the environment's "Editor", the students came across the fact that there was no in-built MoPiX symbol (such as the " $x$ ", " $V x$ ", that respectively represent the position and the velocity in the $x$-axis) to express the idea of an object becoming green under certain conditions. The first thing they did so as to overcome this problem was to invent a new symbol that would express a varying quantity. The "gineprasino" (i.e. "become green" in Greek) symbol was decided to represent in the "template" the varying quantity and the " t " variable to be used so as to describe the "at any time instance" aspect. | Creation of the new symbol "gineprasino" ("become green") relates MoPiX language to the original everyday expression of the goal of the activity. This can be considered a new boundary term to be meaningful both in everyday language and in MoPiX programming. <br> This is what they want to happen. |
|  | Having completed the left part of the equation (i.e. the "gineprasino(ME,t)= $\qquad$ ") and in order to complete the right part as well, the students used -as noted before- the $Y$ coordinates of the two objects and the less than sign to relate them. Surprisingly, the way in which they used the $Y$ coordinate concept for each object was completely different. The ball's $Y$ coordinate was expressed in terms of a quantity varying over time (i.e. the " $y(M E, t)$ "), while the ellipse's $Y$ coordinate was expressed | The Equation Editor helps to structure the kind of expression that can be formed. It is not clear what role ME or $t$ might play. They may be included simply by pattern following. Students have already established a difference between the use of specific values and variable $t$ and would be likely to recognise that they do not want to specify a single value. <br> The development of the right hand side of the equation |

in terms of the constant arithmetic value corresponding to the object's at that time position on the Stage (i.e. the " 274 "). Adding the "less than" sign in between, the first equation eventually developed was the "gineprasino(ME, t$)=\mathrm{y}(\mathrm{ME}, \mathrm{t}) \leq 274$ ".

Unexpectedly, this equation didn't cause the ball to become green since it solely described the event to which the ball would respond (being below the ellipse) and not the ball's exact behaviour after the event would have occurred (change its colour). To overcome this obstacle, the students decided to construct another equation in which they tried to find out ways to integrate the "gineprasino" variable. The structure of an equation they had previously used, the $" V x(M E, t)=(\operatorname{not}(a m I H i t t i n g A S i d e(M E, t-1)) \times(V x(M E, t-$ 1) + Ax (ME,t-1) $)+\left(\right.$ amlHittingASide(ME,t-1)) $\times(\mathrm{Vx}(\mathrm{ME}, \mathrm{t}-1) \times-1)^{\prime \prime}$ which explains what happens to a ball's velocity when it hits on one of the "Stage's" sides and the way in which the "amIHittingASide(ME,t)" variable deriving from the "amIHittingASide(ME,t)=(x(ME,t) 0 or $\mathrm{x}(\mathrm{ME}, \mathrm{t}) / 799)$ and $\mathrm{Vx}(\mathrm{ME}, \mathrm{t}) \neq 0$ " equation was incorporated in it, were the two elements that the students recognised as helpful in the construction of their second equation. Recognizing the "Vx(ME,t)" equation's similarity to the one they were trying to develop instead of what happens to the velocity under certain circumstances they would determine what happens to the colour-
$y(\mathrm{ME}, \mathrm{t}) \leq 274$ is not surprising to me!
"gineprasino(ME, t$)=\mathrm{y}(\mathrm{ME}, \mathrm{t}) \leq 274$ " is a close translation of "When the $Y$ is something less than 400, it's green" in turn 11. This completes the transition from the everyday expression of the goal to expression of the goal in MoPiX formalism. The structure of the expression remains essentially the same, though the order is reversed to be compatible with the standard MoPiX equation convention.

This stage mirrors the students' previous experience when using "amlHittingGround", which they assumed would have the immediate effect of making the ball hit the ground.
(The relevant data is not included but I assume that at some point they discovered that they needed both this equation and the one defining $V y$ in terms of it in order to make the ball bounce off the ground.)

Having made an association between the structure of gineprasino(ME, t$)=\mathrm{y}(\mathrm{ME}, \mathrm{t}) \leq 274$ and that of the amlHittingGround equation, drawing on the everyday language/ boundary term association already made at turn 2 , they are then able to associate their goal with the previous goal of making the ball bounce that involved using two equations: one to check the condition and one to assign the value.
(Did they actually use amlHittingASide to make their template? This is interesting - a further link in the semiotic chain that allows amIHittingGround and amIHittingASide to substitute for one another.)

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and a similarity between the "amlHittingASide" and "gineprasino"
variable, led students to duplicate this equation's structure,
eliminate any content and use it as a template to designate what
happens to the ball's colour when it is below the ellipse. The
equation to be completed was the "greenColour(ME,t)=
```

$\qquad$

``` which they had used in the past in the form of "greenColour(ME,t)=100" in order to give \(100 \%\) green colour to their objects.
To link the first equation which encompassed a new symbol to the second one which included symbols that were in-built in the MoPiX environment (i.e. the "greenColour"), the students utilized the "gineprasino" variable in a similar way to the "amIHittingASide", exploiting the fact that this variable may receive two distinct values (1 or 0 ) according to the ball's position. To complete the equation, students used two arithmetic values, the " 0 " and the " 100 ", to express the percentage of the green colour the ball would contain in each case (i.e. the ball being above or below the ellipse). Thus, the second equation developed was the: "greenColour(ME,t)=not(gineprasino(ME,t))×0 + gineprasino(ME,t) \(\times 100\) ".
The above episode contains many interesting events that indicate the existence of a qualitative transformation of the students' mathematical experience in reifying equations, as emerged through their interaction with the available tools. These events suggest that the students were able to develop insights into the use of equations' formalism to create and control animated models as well as to cope effectively with structural aspects of
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This leads them to make use of the provided equation as a template.
?? How did they decide that they should use greenColour(ME,t)? (Yes, they have used in the past to allocate green colour to an object, but how did they come to use it at this point? We need more transcript to see this.)

They may be exploiting the $1 / 0$ truth function, but is this a deliberate exploitation or just following the pattern of the template equation? Again we need more transcript to judge this and the use of 0 and 100.

This analysis traces how the expression of the students' goal moves from everyday language to MoPiX formalism by identifying links in a semiotic chain.

The 'boundary terms' - everyday words incorporated into the names of MoPiX functions/variables play an important role in this movement, allowing the selection of relevant equations from the
equations, involved in the making sense of them as objects that underlie the respective models' behaviours.
library to act as templates for creation of the new equations.
Is it significant to this process that the condition chosen by the students - being above or below a given position (turn 1) - is similar to that used for hitting the ground? The question by S2 in turn 2 "How did you define the plane which is the ground?" seems to play a key role in the construction of the semiotic chain. I wonder if the same type of process would have occurred if the condition had been dependent on a different variable, for example, time rather than position.

## Commentary from IOE perspective on the two analyses of Episode 2

The episode is of interest as an example of student construction of new equations, extending the MoPiX lexicon to allow achievement of a specific goal. As indicated in the summary of our theoretical frame above, our reformulation of the common research question for the IOEbased MoPiX teaching experiment focussed on the ways in which students operated with the various modes available to them and the relationships between their semiotic activity and mathematical meanings relevant to the educational goals of the teaching experiment. In approaching this episode from the ETL teaching experiment we adopt the same focus, though in this case the goal is that adopted by the students themselves: the solution of the problem of making a ball change colour as it passes another object. The questions we pose are thus:

> What forms of language and other modes do students use to communicate about their problem solving?

How do the semiotic resources used by students contribute to the problem solving process?

As we are looking at a process of development of a solution over time, we adopt the analytic approach of tracking the connections between the semiotic resources used as the episode progresses, reconstructing a semiotic chain as a means of examining how the everyday description of the visual characteristics of the imagined solution is transformed into a set of MoPiX equations that will realise that solution. The chain is reconstructed by identifying at each step of the episode the source of the semiotic resources used by the students (cf. Carreira, Evans, Lerman, \& Morgan, 2002). In this episode the sources identified include 'everyday' language (e.g. "the ball is above the ellipse", mathematical language (e.g. " $x, y$ coordinates"), MoPiX formalism (e.g. "gineprasino( $M E, t)=y(M E, t) \leq 274$ "). These three types of source are a priori categories used extensively in the analysis of data arising from the IOE experiment. In analysing this episode, we additionally identified what we have called 'boundary terms' such as "ground" which are both part of everyday language and components of pre-existing MoPiX terms. Such boundary terms seem important in the formation of the semiotic chain, as they are both meaningful in the everyday discourse and functional within the MoPiX environment. Indeed, we see the newly formed term "gineprasino" to serve as a boundary term that plays a linking role in the transformation of the students' original vision into a functioning MoPiX model.

Comparing the IOE analysis of this episode with that provided by the ETL team, we note that both teams highlight the students' identification and exploitation of similarities between their current goal and previous MoPiX experience with equations including terms such as "AmIHittingGround". The IOE focus on links in a semiotic chain, however, leads to some different interpretations. In particular, the construction of the equation "gineprasino $(M E, t)=y(M E, t) \leq 274$ ", seen as surprising in the ETL analysis because of the different forms of expression used for the y coordinates of the ball and of the ellipse, is explained in the IOE analysis as a close translation of the students' earlier everyday/ mathematical language expression "When the Y is something less than 400, it's green". The IOE analysis sees structural similarities both between expressions in different semiotic
systems and between expressions within the MoPiX system as playing crucial roles in enabling the students to move towards the solution of their problem. While the ETL analysis also lays importance on links between expressions within MoPiX, in particular the use of existing equations as templates for the construction of new ones, it additionally posits the use of conceptual similarities between existing and new equations. While not denying the possibility of such conceptual connections, the IOE approach does not address them.

## Reflections on the cross-analysis

Differences in the design of the experimentation have been affected both by differences in the theoretical orientations and research interests of the two teams and by differences in the global and local contexts within which they were working. It may be seen that key theoretical issues guided the design of tasks for each team: for IOE, the provision of a multi-semiotic environment and opportunities for communication in a variety of modes; for ETL, the notion of half-baked microworld. In both cases the provision of opportunities for exploration, collaboration and social interaction among students was also a theoretically informed component of the design.

The cultural and institutional contexts of the two experiments had a strong influence on the possibilities available to each research team and underlie the most fundamental differences between pedagogical plans of the two teams. The IOE team works within a national context that is strongly governed by a tightly defined curriculum, regulated by frequent high-stakes examinations. Negotiation of entry into institution in which the experimentation took place required a high degree of compliance with the standard curriculum, although development of new forms of pedagogy was welcomed. The IOE pedagogical plan thus was designed to focus explicitly on agreed topics within the curriculum with educational objectives that could be seen by students and teachers to be relevant to success in examinations. In contrast, although the broader cultural context within which the ETL team is situated may not encourage pedagogical innovation, the team was able to negotiate access to the institution in a way that allowed them freedom to design a programme of work independent of specific goals of the standard curriculum and, indeed, deliberately by-passing conventional practices.

MoPiX has potential to be address concepts that lie at the intersection of mathematics, physics and engineering. The curricula of different countries define the boundaries and relationships between these domains slightly differently. The ETL team report the place of mathematics as separate from science or engineering and describe the design of their activities as unconventional in bringing together the mathematics of equations with the modelling activities of engineering. In contrast, for the IOE team, the standard mathematics curriculum includes applications of mathematics, including 'mechanics' topics that in other countries are considered to be part of physics. This difference in definition of mathematics is reflected in the focus of the activities and the research interests. For the ETL team, the focus of their tasks was on the construction of models using mathematics and their research interest concerned the use of mathematical formalism. In contrast, for the IOE team, the development of the concepts of velocity and acceleration was an important curricular objective and, within a general research interest in student use of multiple modes of representation, they focussed specifically on the ways students represented these concepts.

The analytic approaches taken by the two teams are not on the whole incompatible and yield interpretations of the data that have some similarities. For example, in analysing episode 1, both analyses consider interaction between the symbolic formalism of MoPiX and its graphical effects as critical to the problem solving process and identify the student's final inspection of the two sets of equations of the two models as key to his making connections between velocity ad acceleration. However, the two teams emphasise different aspects of student use of the representations used by students and a significant difference may be identified in the two analyses in the ways in which the two teams treat the representations offered by MoPiX and the relationships between different systems of representation. For ETL, the symbolic formalism of MoPiX plays a particularly important role because of the way that it provides access to the 'deep structure' of the environment. While the IOE perspective does not deny this role of symbolism, it does not play an important part in the IOE analysis. Rather, the various systems of representation are considered of equal significance, the interest being in analysing what each brings, both individually and in combination, to the possibilities for meaning making and problem solving. This difference is highlighted in the interpretations of the equation gineprasino(ME, t$)=\mathrm{y}(\mathrm{ME}, \mathrm{t}) \leq 274$ in episode 2 . Whereas the ETL analysis focuses within the symbolism on the different ways in which the value of the y coordinate is expressed, the IOE analysis attends to the relationship between the structure of the equation and of students' everyday expression of their goal.

## IV.6. Potential offered for the theoretical landscape

- How is meaning making perceived?

The constructionist theoretical framework informs both the original design of MoPiX and the pedagogic design and research of the ETL team. However, constructionism is not perceived by ETL as an individualistic theory that views the meaning generation process as being detached from the students' interactions with their social environment and as a synonym to the cognitive development, achieved merely through the individual's interactions with the given computational tools. This is evident by the fact that ETL chose to use as a key theoretical construct for the task and research design, the notion of half-baked microworlds.

The half-baked microworlds are by nature designed for instrumentalisation since they invite the students to deconstruct it, change it and build on its parts, engaging in the way in exploration and construction activities, rich in the generation of meanings. However, they are also perceived as boundary objects, in the sense that they can convey meaning among of the members of the same community, operating as a tool of communication, around which the members of the community organise their activities. Bearing in mind this particular aspect, ETL designed tasks that allowed students to participate in different social orchestrations within their class discussing and sharing artefacts and engage in intrumentalization processes in a collaborative context, in different social levels (within their workgroups and within the whole class). Thus, the generation of meanings is considered to emerge and shaped both by the student's mathematical activity as they interact with the available tool (the microworld and its representations) and their social activity supported the half-baked microworld's own characteristics.

From a social semiotic perspective, meaning is not an individual but a social phenomenon, created and existing in interaction through the choice and deployment of semiotic resources. Possible meanings will depend on the context of the interaction, including the prior experience of the individuals involved, as well as the semiotic resources available. However, this conception of meaning does not allow direct statements to be made about the cognition of individuals or about learning. In order to consider learning it is necessary to define this in terms of change in patterns of interaction.

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## V. MaLT cross-case analysis

## V.1. Identification

ETL - Educational Technology Lab, Education Department, PPP Faculty, University of Athens<br>IOE - Institute of Education, University of London/ London Knowledge Lab

## V.2. Contextual elements

## ETL context

The implementation of the familiar MaLT pedagogical plan took place in the computer laboratory of a multi-cultural secondary school in Athens (2nd Multicultural school of Helliniko) with one 7th grade classroom ( 13 years olds). The entry to schools for classroom research within the typical centralized educational system of Greece is very difficult. Most of the researches that have been officially permitted by the Pedagogical Institute (the legal organisation that has the responsibility for that) to take place in schools are based on the use of questionnaires. Classroom-based research interventions which cause 'perturbation' in normal educational life are not easily implemented in schools due to the fact that the curricular time limits are rather limited.

At the lower secondary level the teaching of mathematics takes place for four teaching sessions ( 45 minutes) a week. The school time schedule, the content and the curricular goals are determined by the National Curriculum. Although computer use for the teaching of mathematics is not officially part of the curriculum, the National Curriculum suggests the use of particular computational environments (e.g. Cabri, Function Probe) for the teaching of specific mathematical topics. However, very few teachers follow these suggestions for two main reasons: (a) most of the time computer laboratories are occupied due to the teaching of informatics and (b) teacher training for the use of computers in the teaching of mathematics is rather limited.

Due to the above reasons, the use of computers in mathematics is not concerned with the normal school practice and thus it can be conceived as an innovation within the Greek educational system. As mentioned above, however, the system officially gives space for teachers to enrich their lessons with activities in the computer laboratory. In the last years, the Ministry of Education, in the frame of a general attempt to introduce a cross-thematic approach to learning, has implemented in specific lower secondary education schools, the "Flexible Zone" program. The "Flexible Zone" aims at linking horizontally the content of all subjects taught in school by introducing a two-hour per week session during which students engage in cross-curricular projects and activities. These projects give students the opportunity collaborate with their peers through their engagement in exploratory activities, often based on the use of computers.

The school in which the ETL experiment took place had fully integrated the "Flexible Zone" in its regular weekly timetable. The access to the school was negotiated through a mathematics teacher which we personally knew as she had attended a year before the

Mathematics Education postgraduate course at the University of Athens. The teacher brought us into contact with the school's head-master who strongly supported the use of digital media in the "Flexible Zone" program. This was a usual practice of our team. In order to by-pass official or other constraints posed by the educational system we used our personal contacts with teachers or head-masters (e.g. PhD candidates, participants in teacher training courses at the university) so as to support the implementation of design experiments involving innovative uses of computational tools in real classroom settings as a means to generate and enhance meaning-making situations.

The school teacher did not participate in the implementation of the activities in the classroom due to other commitments. However, she participated in three preparatory meetings with the researchers at the school and contributed to the focus of the activities in relation to students' mathematical background. She also provided information for the students' performance in mathematics and possible difficulties that they might face when engaged in using MaLT.

The fact that the lesson was part of the "Flexible Zone" allowed the ETL team not emphasise on closed 'didactical goals' but, rather, attempt to challenge students' construction of meanings for angle when they were introduced with new visual representations of angular relationships in 3D space within MaLT.

Teaching was conducted by a member of the ETL team who acted also as a researcher in the classroom. In each session a second teacher - also member of our research team- supported data collection acting as co-researcher. The class had totally 18 teaching sessions over two months. Each session took place in the computer laboratory of the school. This room was equipped with PCs around the periphery where students sat around tables working in groups.

## IOE context

The IOE (alien) teaching experiment with MaLT took place in a voluntary aided Church of England secondary school in South London. Although located in a middle class neighbourhood, the school population is drawn from a much wider area and includes a significant proportion of students from low-income families and with African and AfricanCaribbean backgrounds. The school was only opened a few years ago and at the time of the teaching experiment contained pupils in Years 7-11 only (aged 11-16 years). As a new school, it is well equipped with computers, both laptops for use in ordinary classrooms and desktop PCs located in laboratories. However, the mathematics department does not habitually make use of these facilities. The school ethos is 'traditional', with a strong emphasis on academic achievement and tightly constrained student behaviour. Within the mathematics department, the dominant form of pedagogy is teacher-led, using the 'three-part lesson' structure recommended by the National Strategy (DfES, 2001).

Entry into the school was negotiated through contact with the Head of the Mathematics Department, a graduate of the Masters programme in Mathematics Education at the IOE, and another member of the mathematics teaching staff (GD) who had attended research seminars in mathematics education. These members of school staff gained permission from the Headteacher of the school, who was happy to be involved with a university-led project and to
be seen to encourage innovation. A condition of this permission, however, was that the teaching experiment should not prejudice student performance in national tests. This condition had two effects on the conduct of the teaching experiment. First, the pedagogical plan had to be designed to match the official curriculum. Second, the class chosen to take part was a 'low set' that was not expected to do well in the national tests. Each year group in the school was divided into teaching 'sets' according to their performance in previous years. Although studying the same National Curriculum, 'lower' sets may be expected to cover less material in less depth.

The class included 24 Year 8 students (aged 12-13). As set 4 out of 5 in their year group, their levels of attainment were significantly below average. Students from lower income families and non-white backgrounds were over-represented in the class, relative to their numbers in the school as a whole. Their experience of school may be described as marginalised as they were seen by teachers, by other students and by themselves as unable to achieve the academic standards valued in the school. As marginalised members of the community, the behaviour of many members of the class was also non-conformist. Most were poorly motivated and displayed a high degree of off-task and, in some cases, disruptive behaviour. The class teacher's usual pedagogy was strongly framed, allowing little space for students to deviate from set tasks or to interact with each other. Her classroom was normally arranged with tables arranged so that students could sit in pairs facing the front of the room. For the parts of the teaching experiment that took place in her classroom, these tables were rearranged to allow groups of 4-6 students to work together. This way of working was unfamiliar to students within mathematics lessons and needed frequent intervention to enable constructive interactions to take place. The parts of the teaching experiment involving use of MaLT were located in a computer laboratory with sufficient PCs for every student. Although we wished students to work in pairs with MaLT, their prior experience of being in the computer laboratory (mostly for IT lessons) had involved only individual work and they preferred to work individually, resisting attempts to get them to share PCs.

The class teacher was keen to participate in the project, recognising that this was a substantial innovation compared to her usual practice. She participated in the planning and played a role in team teaching during the implementation. A student teacher, who was undertaking a work placement in the school, also participated in team teaching, together with two IOE researchers who both acted as teachers and gathered data.

The pedagogical plan was shaped by the official curriculum and by reflection on what might motivate both the students and the teachers involved as well as by the interests of the IOE research team. It was decided to set it in the context of a project contextualised by a 'real problem', partly because it was felt that this would be of interest to the students and partly because this type of project is an officially recommended approach to teaching students of this age in the UK. In order to 'cover' the official curriculum it was necessary to include paper-and-pencil work as well as work with MaLT. The time allocated to 3D topics within the official curriculum was less than we required for the project. The class teacher agreed to be somewhat flexible in her following of the school's scheme of work and also arranged two after school sessions additional to normal classes.

| NKUA/ETL | IoE/LKL |
| :---: | :---: |
| Secondary school | Secondary school (Church of England) |
| 1 class of 20 students | 1 class of 24 students (low attaining) |
| 7th grade pupils, 13 years old | Year 8 students, (12-13 years old) |
| Workgroups of 2, each with a Tablet PC | a) groups of 4-6 students in the normal classroom <br> b) individuals, each with a PC, in a computer laboratory |
| 18 teaching sessions, 2 months | 9 teaching sessions, 2 weeks ( 4 sessions using MaLT in the computer laboratory) |
| Interactions within and between workgroups | Interactions within and between workgroups |
| One ETL researcher as experimenting teacher (responsibility of the lesson), a second researcher supporting data collection (co-researcher) | Team teaching involving class teacher, student teacher and two IOE researchers. |

Table 1: Comparison of contextual elements.

## V.3. Theoretical frames

## ETL theoretical frame

The main theoretical frame adopted by the ETL team is constructionism (Papert, 1980, Harel and Papert, 1991, Kafai and Resnick, 1996). Based on this theoretical origin we draw on the idea of teaching and learning mathematics with the use of technology with learners as central sense-making agents while interacting with specially designed exploratory computational tools and representations viewed as integral to mathematical activity rather than an external aid to internal cognitive processes. The constructionist framework expects students to interact
with and manipulate the representations provided by the tool, making sense of their behaviours through this interaction with the computer environment and with the social context of the classroom. MaLT is a microworld environment for geometrical constructions which is designed to provide multiple linked representations by combining symbolic notation -through a specially designed version of Logo- with dynamic manipulation of graphically represented mathematical objects.

In designing the pedagogical plan and the teaching experiment we took into account Papert's view (1980) that learning environments based on the use of dynamic digital tools are much richer in opportunities for generating meanings. We intended to move in the direction of identifying tools and tasks to facilitate students' meaningful engagement in exploring the angle embedded in simulations of familiar situations in 3D space. The constructionist theoretical perspective of the MaLT pedagogical plan was based on the assumption that programmable geometrical constructions designed to help children abstract the notion of turtle movement in the 3D space provide a useful environment for developing their conceptualizations of geometrical objects in 3D space, like angles.

Thus, the wording of our reformulation of the Common Research Question by our team specified the priority of the student's engagement in experimenting with the available tools by introducing a distinction between the term 'representations' (which existed in the general form of the CRQ) with the students' use of representations:

How do student use the available representations in MaLT to construct meanings for the concept of angle in $3 D$ space

- as a geometric shape, i.e. formed between two geometrical objects which can be segments in 2D geometrical figures (e.g. rectangles) or 2D geometrical figures in the $3 D$ space (e.g. dihedral angles);
- as a dynamic amount, indicating a change of directions which can also be represented by a variable;
- as a measure represented by a number.

Under a constructionist perspective, the phrase "to construct meanings" indicates the dialectic relationship between action and meaning implying that within the activities students were expected to experiment with different strategies and, more importantly, attach personal meanings to the results of their activities. In this view, computational tools provide a system through which mathematics can be expressed. Thus, they orient students toward a mathematical perspective which can be traced when students use them to develop an explicit appreciation of relations (i.e. the relational invariants) and their semantics (i.e. the meanings). The multiplicity of roles that tools play suggests a detailed analysis of student's thinking-inchange in order to capture the subtle shifts in meaning generation and how these might have been mediated by the use of the available tools.

In analysing students' mathematical thinking we were interested to capture the ways in which the students interacted with the available representations and the ways in which the meanings
they constructed structured and were structured by them. This is what the notion of situated abstraction (Noss \& Hoyles, 1996) seeks to address, i.e. to describe how learners construct mathematical ideas drawn on the linguistic and conceptual resources available for expressing them in a particular computational setting as well as the ways in which learners exploit the available tools to move the focus of their attention onto new objects and relationships. Yet, from a social constructivist perspective, psychological and social aspects of learning can never be considered separately and the term situated abstraction captures the synergy between them: student's activity within a community (Lave \& Wenger, 1991) both shapes and is shaped by their interaction with the available tools and those around them. The idea is that students could web their own thinking by communicating with and through the tools of the microworld and shaping them to fit their own purposes, including the need to communicate with others. In our view, situated abstraction can be seen as a complementary -physical and intellectual- context providing new meanings, new resources for the learners to (re)think-inprogress while exploiting the available tools to move the focus of their attention onto new objects and relationships.

Angles are critical to 3D geometrical knowledge and since they are related to students' everyday experience they can be considered as a rich domain for mathematics meaningmaking, not systematically studied up to now. It is also clear from previous research that students have great difficulty in coordinating the various facets of the angle embedded in various physical angle contexts involving slopes, turns, intersections, corners, bends, directions and openings (Mitchelmore \& White, 1998). Taking this into account Mitchelmore and White (2000) highlight that a mature abstract angle concept depends essentially on learning to link the various physical angle contexts together through "the systematic attempt to investigate our spatial environment mathematically" (p. 233). Indeed, the teaching of 3D geometrical concepts is an area of mathematics in which students' informal ways of experiencing the physical 3D space around them are excluded by the teaching approaches in the school. In activities involved in the familiar pedagogical plan provides students with challenges to construct, transform and animate 3D geometrical objects often encountered in everyday physical angle situations - such as doors and revolving doors. For example, in a 'door' simulation the arms of the defined dihedral angle are rectangles while in a 'spiral staircase' simulation the arms of the defined angle are an equilateral triangle and a rectangle. Thus, the tasks are designed to integrate different angle domains (e.g. intersecting, turning, sloping) related to physical angle experiences in everyday circumstances (e. g. corner, slope and turn) as well as to the main definitions of angle.

In considering specific mathematical aspects of students' activities at the level of design and analysis our research perspective was also informed by the theoretical construct of conceptual field (Vergnaud, 1990). In the light of this construct, it makes no sense to perceive angle in 3D space as a mathematical notion on its own but rather it is more useful to consider it in terms of the concepts interrelated with it, the situations in which it may be used and the available representations. For instance, a concept tightly related to angle in 3D space is that of a turn, a situation in which it may be used can be a situation evoked by a given task (e.g. an
opening-closing door simulation) while the available representations can be based on the use of paper and pencil or on the use of computational tools.

## IOE theoretical frame

The primary theoretical framework adopted by the IOE team involved in the teaching experiment is multimodal social semiotics (Kress, Jewitt, Ogborn, \& Tsatsarelis, 2001; Kress \& van Leeuwen, 2001; O'Halloran, 2005). This informed the design the teaching experiment and the analysis of its results. Although originating in linguistics, this theoretical framework challenges the primacy of language as a means of communication and meaning making, highlighting the different potentials for meaning offered by different modes of communication. Multi-modal and multi-semiotic environments allow participants many opportunities for making meanings with the representations available to them and choices about the most apt representations to employ in order to communicate their desired meanings.

In designing the pedagogical plan we saw MaLT as an environment in which students could experience new kinds of representation of 3D objects. 3D geometry is an area of the curriculum that appears difficult both to learn and to teach. 2D representations of 3D objects are frequently used in order to support analysis of their geometrical features, yet the success of this approach relies on students' ability to make connections between the different representations. In practice, the focus of the UK curriculum is often more on developing skills in constructing particular type of 2D representations rather than on using them in order to develop understanding of the 3D objects themselves. We aimed therefore to engage students in using a range of representations purposefully and making connections between them in order to gain a fuller understanding of 3D geometrical objects. The pedagogic plan took the form of a project based on a 'real world' context that would also have meaning for students within discourses from outside the classroom. They would thus be likely to draw on everyday discourses and forms of representation as well as on the formal mathematical discourses and representations encountered in the classroom. This was designed to provide opportunities for them to form links between these discourses, enabling sense to be made of the representations that are new to them. We understand such links between different domains of activity (sometimes referred to as 'transfer') to occur through the formation of chains of signification, where similar signifiers are encountered in different discourses (Carreira, et al, 2002).

Through the course of the set of activities, it was intended that students would make use of a range of semiotic systems, both paper-based and in MaLT, visual and symbolic, with different elements and grammars. Each of these semiotic systems has a different meaning potential (O'Halloran, 2005; Kress, 2001). Thus making use of one of the systems, for example, isometric drawings, offers a particular set of opportunities for developing understandings of the properties of 3D shapes, while another, such as nets, offers different opportunities. The symbolic logo-based programming of MaLT provides a further semiotic system that makes more explicit use of angle and length relationships within figures than most paper and pencil based systems.

The juxtaposition of multiple semiotic systems thus provides a rich environment for developing understanding of 3D geometrical objects. Operating separately with the systems
ensures that students encounter different aspects of the properties of such objects. Most importantly, however, operating with more than one system provides important opportunities for developing a fuller understanding of these properties and of the relationships between them. In considering specifically mathematical aspects of multimodality, we refer to Duval (2006), who argues (from a different semiotic tradition) that conversion between semiotic systems (which he names representational systems or 'registers') is of fundamental importance to mathematical learning. Conversion demands that the student distinguishes what is mathematically relevant in each system and separates the mathematical object from its representation. The computational environment provided by MaLT not only juxtaposes different semiotic systems for representation of 3D objects but also, by making one system depend on another, seems likely to facilitate 'conversion' and consequent abstraction of mathematical properties. It not only demands that students engage in using different forms of representation for the 'same' mathematical object but also that they actively use the representations provided by the Logo symbolic programming and by the dynamic variation tools to effect changes in the visual form of 3D representation in the turtle screen. In the opposite direction, the process of 'debugging' faulty 3D figures again demands conversion: identifying those parts of a Logo script responsible for the 'buggy' behaviour. According to Duval's principle, we hypothesised that activities in this environment would enable students to develop their understandings of the properties of 3D shapes. This hypothesis, however, did not form a focus for our research as our overarching social semiotic perspective directed our attention to communication and meaning making in social contexts rather than to investigation of individual cognitive development.

Social semiotic analysis of communication is not concerned with the cognition underpinning the communication or with the intentions of those involved but with the meanings produced in the social interaction. Our reformulation of the common research question thus focussed on the ways in which students operated with the various modes available to them and the relationships between their semiotic activity and mathematical meanings relevant to the educational goals of the teaching experiment:

- What meanings do students make in relation to three dimensional geometry through their semiotic activity in the context of working with MachineLab and other modes?
- What relationships are there between procedures students write or changes they make to given procedures and the properties of the shapes they are working with?
- How do students use the variation tool and for what purpose?
- What interpretations do students make of the effects of using the variation tool?
- What choices do students make between and within semiotic systems in order to communicate their completed design to their peers? To what extent and in which ways are the properties of shapes represented?
- To what extent are students' constructions in different semiotic systems consistent with one another? In particular, are representations of properties of shapes consistent in different systems?

The data collected and the methods of analysis also reflected this perspective, drawing together student productions in different modes (paper-and-pencil, MaLT, speech, gesture) and tracing both the meanings produced through each mode of communication and the interactions between modes.

## V.4. Comparison of didactical functionalities

Differences in the educational goals of the two teaching experiments, the different contexts of implementation -based on different didactic cultures- and in their theoretical frameworks led the two research teams to focus on different sets of didactical functionalities (see Table 2). The one aspect that was clearly significant for both teams was related to the interpretation and use of the two new kinds of turtle turns in MaLT (uppitch/downpitch, leftroll/rightroll) which are strongly related with the passage from one plane to another in 3D space. This characteristic of the DDA was salient for both teams and seemed to serve a similar function for both in relation to students making sense of (a) angle as a turn indicating both the act of body turning and the result of it, which inevitably involves directionality (dynamic scheme) and (b) angle as a turn represented by a number (measure scheme).

The other didactical functionalities highlighted by the ETL teaching experiment focus on the role of specific representations provided by MaLT and their role in students' conceptual understanding and struggles in making sense of angle in 3D space indicating the potential of them for expressing and reflecting on the mathematical nature of angle as a spatial visualisation concept.

Students' experience with MaLT in the IOE teaching experiment was limited both by the time constraints of the official curriculum and by the students' own insecure prior knowledge of angle. The major focus of their work was thus the struggle to make sense of and use the Logo language and to relate it on the one hand to the representation in the turtle screen and on the other to their physical experience of 3d space.

|  | characteristics of the DDA | educational goals | modalities of use |
| :---: | :---: | :---: | :---: |
| Both IOE and ETL | Ther new commands of turtle turns and the visualisation of the turtle while executing them. | The new kinds of turtle turns allowed students get acquaint with the navigation of the turtle in 3D geometrical space of MaLT environment. | The navigation of the turtle in 3D space facilitated students' engagement in <br> (a) using gestures and/or objects to make sense of the turtle's move/turning in 3D space <br> (b) linking the concept of angle |


|  |  |  | as a turn with particular measure and that of angle as a slope <br> (b) oscillating between different frames of reference for interpreting the move/turning of the turtle in 3D space. |
| :---: | :---: | :---: | :---: |
| ETL | The use of dynamic manipulation tools of MaLT support a conception of angle as dynamic spatial concept based on turn and directionality. | The dynamic manipulation with the use of the Unidimensional Variation Tool (1dVT) provided students with an alternative representation, enabling a new approach to angle in 3D space based on the innovative use of familiar mathematical representations (e.g. number line). | The dynamic manipulation of variables allowed students to construct meanings for angle as dynamic entity for moving in different planes. This was also facilitated the investigation of the role of 2D representations (i.e. 2D geometrical figures) in forming angular relationships in 3D space mainly through the simulation of 3D objects that involve turning (e.g. door). |
| ETL | The graphical representation of geometrical objects can challenge students' conceptions of the conventions used to represent 3D object in the computer screen. | Challenge pupils to move the focus of their attention from directed turns between lines to directed turns between planes defined by geometrical figures. | However, students' investigations were affected by the interaction with the other representations available. Specifically, (a) symbolic notation provided a basis for identifying the role of variables in generating specific simulations involving continuous turning and (b) dynamic manipulation through the use of the 1dVT enabled the dynamic move of a geometrical object in 3D space and thus the observation of it from different perspectives. |


| IOE | Symbolic representation the Logo editor. | The need to provide a set of formal instructions to construct a 3D figure is intended direct attention to the formal properties of the figure (angles and lengths) and relatinships between different parts of a figure. | In practice we found that students were reluctant to adopt an analytical approach, preferring to either use preformulated sets of commands or to use trial and error. A common strategy to find the required turtle turn was to enter a turn command and then pressing the Insert key repeatedly until the turtle appeared to be in the correct position. |
| :---: | :---: | :---: | :---: |

Table 2: Didactical Functionalities.

## V.5. Results of the cross-case analysis together with illustrative examples

In the context of the cross-analysis we have chosen to provide feedback on the designed pedagogical plans by addressing contextual issues as the IOE team designed a different pedagogical plan to fit its cultural context but incorporated tasks based on those developed by the ETL team. As far as the analysis is concerned we will attempt to provide converging interpretations of students' behaviors in both experiments.

## Inside-outside the Turtle Scene and the use of gestures ETL analysis

The issue 'inside-outside the TS' emerged for the ETL team as one aspect of the analysis concerning the student's learning trajectories and potential difficulties in coordinating different aspects of the concept of angle when navigating the turtle in 3d space. ETL researchers were interested to explore the role of the body-syntonic metaphor in students' use of the available representations in MaLT as well as its influence on the students' construction of meanings for angle in 3d space.

The ETL team divided the activity sequence in two phases and developed for each one of them a strand of two tasks. After a short familiarisation phase with the basic Logo commands, in task 1 of the phase 1 students were asked to move the turtle in the right and left corner of the 3d TS and then to bring it back at its initial position. It was not specified on purpose what was meant by 'the left and right corner', so as to leave students to explore 3d space and develop their own navigation strategies.

During their engagement with task 1 , most of the pupils seemed to prefer to work exclusively on the horizontal plane indicating the initial position of the turtle, which was conceived by the students as the 'ground plane' (although it is actually a horizontal one parallel to the 'flagged
plane' visualised at the bottom of the scene). Despite the researcher's request to navigate the turtle at different corners of the 3 d scene, most of the students have chosen to move the turtle on the 'ground plane' so as to construct either simple crooked lines or familiar geometrical figures such as rectangles. At this stage of their work students seemed to image that they were 'inside' the scene driving the turtle in a body-syntonic way by projecting the orientation of their trunk to the orientation to the turtle. In other words, students seemed to work as if they were inside a 'room' navigating the turtle to design geometrical figures 'on the floor'. Pupils' experimentation concerning the notion of angle in this phase was developed around how to find out the measure of turtle turns for the construction of 2 d figures without an explicit reference to the changes of planes, let alone how these might be related to the available commands.

Students' move in other different planes of MaLT appeared during the implementation of the next task of the phase 1 according to which students were asked to simulate the take-off and landing of an aircraft. This task seemed to have provided a more fruitful context for the experimentation of students with both new types of turtle turns involving also angle as a slope, an aspect of angle difficultly recognised by students as the one supporting edge is missing (Mitchelmore \& White, 2000). The analysis at this phase signalled pupils' shift from driving the turtle inside the TS (i.e. to move on a specific 2d plane) to viewing the turtle from a distance (i.e. outside the TS) and drive it according to different frames of reference each time. More particularly, the findings revealed indications that sometimes students followed contradicted frames of reference, focusing for example on the angles drawn in relation to the line of the horizon as visualised in the TS rather than in relation to the previous position of the turtle, as it was the case.

This is evident in the following example which reveals students' confusion over the way in which the commands up(45) and $\operatorname{lt}(50)$ affected/determined the generated graphical outcome visualized in Figure 1. These students (group A) were reflecting upon the commands given to the turtle so as to explain why the aircraft collided to the ground.

Researcher: Hey, nice take off!! I see you hit the ground!
Student: Look there is a slope up(45) and then a slope of $\operatorname{lt}(50)$.


Figure 1: Simulating the taking-off and flying of an $\operatorname{lt}(50)$ aircraft.
$\mathrm{fd}(6) \quad$ It seems that they focus in both cases to the angles drawn in relation to the line of horizon and not in relation to the previous position of the turtle, as it is up(45) the case. A more detailed analysis of pupil's interactions revealed that students oscillated between two different frames of reference:
$\mathrm{fd}(5)$
(a) world frame: defined in terms of directions 'up' and 'down' and
(b) a vehicle frame: typically associated with the orientation of a moving entity, here the turtle.

Though at the initial position of the turtle the 'vehicle' frame of reference coincides with the 'world' frame of reference the use of roll turns might result to contradict one another. In the initial 'take-off' of the turtle the 'vehicle' frame of reference coincides with the 'world' frame of reference. In other words, the 'up' in relation to the turtle's position coincides with the 'up' of the simulated 3d space. Then and especially after the command $\operatorname{lr}(30)$ the two frames contradict one another. However, the students - possibly by drawing upon their everyday experiences - seemed to consider the horizontal ground plane and the directions of up and down as fixed. This may be a possible explanation of their insistence to use the up command in order to get height regardless of the opposite graphical outcome. Thus, ETL team argues that although 3d simulated space is closer to real life and every-day experiences, the bodysyntonic frame which is inextricably linked with the 'world' frame in real 3d space, should be shrunk in favour of the 'vehicle frame' underlying the turtle move in the 3d space.

In this context, the ETL analysis revealed that one aspect underlying students' bodily engagement with the tasks was related to the students' informal or spontaneous use of gestures. It constitutes one aspect of the ETL analysis which emerged as a coherent part of the students' construction of meanings for angle in 3d space interrelated with their attempts to describe the turtle's navigation in 3 d space as well as to conceptualise the role of 2 d representations in forming angular relationships in 3d space. In the next episode the students' use of gestures appears as part of their struggle to understand the ways by which the combination of the new turning commands could affect the manipulation of a 2 d geometrical figure so as to construct the simulation of a door. We note that the episode took place before students were asked by the researchers to experiment with the door simulation which constitutes one activity of the ETL pedagogical plan.

Initially group B constructed a rectangle with three variables on the 'ground plane' (Table 3, Procedure 1). Having recognised the way in which the up(90) command affected the position and the orientation of the turtle, they inserted the command up(90) at the beginning of the respective procedure and constructed the same rectangle on the 'screen plane' which -in mathematical terms- is perpendicular to the 'ground plane' (Table 3, Procedure 2).

| ```to rect :a :b :c fd(:a) rt(:c) fd(:b) rt(:c) fd(:a) rt(:c) fd(:b) rt(:c) end``` | ```to rect :a :b :c up(90) fd(:a) rt(:c) fd(:b) rt(:c) fd(:a) rt(:c) fd(:b) rt(:c) end``` | ```to rect :a :b :c :d up(:d) fd(:a) rt(:c) fd(:b) rt(:c) fd(:a) rt(:c) fd(:b) rt(:c) end``` |
| :---: | :---: | :---: |
| Procedure 1 | Procedure 2 | Procedure 3 |
| Table 3: Logo procedures for the construction of a rectangle in TS (Group B). |  |  |

When trying to concretise the new position of the rectangle in 3d space one students (S2) used her hands so as to mimic the movement of the turtle from the surface to the 'screen plane' (Figure 2).


Figure 2

R: What happened to the turtle with up(90)?
S2: [Showing with her hand the move from the surface to the screen plane] It [i.e. the turtle] took it [i.e. the rectangle] that way.

R: If we didn't put 90 but put 45, what would have happened?
S2: It [i.e. the rectangle] would be nearly in the middle.
R: If we put 50;
S2: Ok, not in the middle. [Showing with her hand] A bit more than that.
The dynamic character of student's bodily engagement in the simulation challenged both of them to try to visualize it on the screen.


Figure 3

So, S1 afterwards had the idea to replace the value 90 in the command $\operatorname{up}(90)$ with a new variable :d to see what would happen (Procedure 3). Dragging on the slider of the variable (:d) in 1dVT had the effect of the figure dynamically moving upwards - downwards visualising in that way the dynamic move of the rectangle in different planes as well as the preceding uppitch-downpitch gestures made by S2 (Figure 3).

S1: [Moving the slider (:d)] Look! If we move it [i.e. the slider (:d)] upwards it [i.e. the rectangle] raises ... If we move it [i.e. the slider] downwards it [i.e. the rectangle] descends.

This episode indicates the conceptualisation of angle as a dynamic entity interrelated with the move to the different planes in the 3d space and the simultaneous visualisation of this move 'inside' the TS. This kind of activity appeared to provide a fruitful domain that challenged student's intuitions and ideas about angle as a spatial quantity come into play interrelated with a dynamic passage from one plane to another.

In the evolution of the episode the students had the idea to insert in the procedure a roll command so as to simulate the continuous move of a door. The sequence of what happened next is as follows. Initially one of the students substituted the command up(:d) (Procedure 3) with one of the roll commands (rr :d). Moving the slider (:d) then she realized that the direction of the axis of rotation was perpendicular to the screen plane ('it turns as a wheel' she said) (Figure 4, on the left).


Figure 4
At his phase students continued to 'play turtle' to identify the type and the sequence of the turtle turns which would result in the desired simulation. In doing so, they faced difficulties in imagining -and thus mimicking- in which way the turtle 'moves' the rectangle in different positions and directions in 3d space. So, they found efficient to rehearse the move of the rectangle with the use of a concrete 3d object, in this case a video cassette, so as to visualise the change of planes of the rectangle as a result of the change of the initial position of the turtle in 3d space (Figure 4, in the middle). So, students realized that initially the rectangle needs to be raised up and then turned ('rolled') on the right. Modifying accordingly the Procedure 3 students used one more variable in the command $\operatorname{rr}(: e)$ that was inserted after the initial command up(:d). They subsequently achieved to simulate the 'opening-closing' door (Figure 4, on the right) by dragging the slider (:e) on the 1 dVT after selecting the value 90 for the slider (:d).

S1: We changed this roll. When we put it [i. e. the roll command $]$ at the beginning [i. e. of the Procedure 3] it [i. e. the rectangle] moves like that [Shows with the cassette the move of the left Figure 5]. So, initially we put it [i.e. the rectangle] in the vertical position.

We can see a dynamic aspect in students' bodily engagement in the episode. While they are to some extent 'playing turtle' with the use of hands and/or the cassette, they define the dynamic manipulation of the rectangle by using position and heading of the turtle which seems to 'coincide' with the rectangle (i.e. the turtle appears in some way to 'carry' the rectangle). The use of the turn and pitch/roll gestures in the above episode supported students' move into
experimenting further with the use variables to represent geometrical objects and the dynamic manipulation of them with the 1 dVT .

## IOE analysis

As the IOE team started to view the video data collected during use of MaLT, it was noticeable that the teachers and researchers made extensive use of gestures in an apparent attempt to support students' planning and execution of constructions in MaLT. One significant type of gesture was a set of stereotyped hand and/or arm movements, often associated with use of the terms turn, pitch (or more frequently up or down) and roll and the associated Logo instructions (see Figure 5 for the codes used in transcription of these gestures). This set of gestures constitutes a new semiotic system, linked with, but not identical to, both the linguistic description of three-dimensional movement and the symbolic system of Logo. Students also made use of these and other gestures to support their communication about turtle movement. We became interested in students' adoption of these new signs and in the relationships between the semiotic activity of teachers and researchers and that of the students.

For the teachers and researchers, using these gestures as ways of thinking and communicating about movement of the turtle within MaLT seemed a natural consequence of our experience with using two-dimensional versions of Logo. The metaphor of 'playing turtle',


Figure5: MaLT gesture codes. an operationalisation of body syntonicity, formed part of our experience of 'Logo culture' and constituted for us a more or less implicit theory about learning with Logo.

In the 3D context, it is not possible to physically act out turtle movements with the whole body. Instead, the hand (or a toy aeroplane held in the hand) substitutes for the body. In planning the introduction to MaLT for the London students, we adopted an initial activity similar to that used by the ETL team, using a model aeroplane to simulate a 'take-off' path and then using 3D Logo commands to reproduce this path in MaLT. In doing so, we found that the turns of the model plane held in a hand became transformed into the set of hand gestures described above. These gestures became incorporated into our further communications about three-dimensional movement throughout the teaching experiment, both spontaneously and as part of deliberate attempts to encourage students to associate a sense of their bodily movement with the Logo symbolism.

We now briefly present two episodes from the teaching experiment in which the teachers and researchers modelled use of gestures to 'play turtle'. Then we present in more detail an analysis of a third episode of a student's use of similar gestures.

## Episode 1:

In the introductory session with MaLT, one of the research team introduced the notion of turtle movement using a toy aeroplane. Holding the aeroplane in her hand, she asked students to instruct her how to move her hand in order to simulate the plane taking off. As she followed the students' instructions, she accompanied the physical movement of the hand/aeroplane with a verbal description, using and stressing the terms pitch, roll and turn in synchrony with the associated gestures.

## Episode 2:

In a later lesson, recognising that some students were still having difficulty distinguishing between these different kinds of turn, the class teacher used her arm and hand to act out the role of the turtle drawing a 'door' under instruction from the class while a researcher entered the Logo instructions into a computer, displaying the resulting turtle path on a large screen. The teacher was careful to follow the conventions of the gesture system in order to emphasise the relative nature of turtle movement. Thus, for example, she turned her hand in a down pitch gesture when given the instruction to go down, even though this resulted in her hand pointing horizontally as in Figure 2. This resulted in conflict for students between their intended outcome and the visual feedback provided, leading to rapid self-


Figure 6: down pitch. correction of the Logo instructions.

## Episode 3:

Student T, having constructed one rectangular wall, was trying to construct a second wall perpendicular to the first. She explained what she was trying to draw using language and gesture. Her words are shown in Table 1, together with a verbal description and a sketch of the accompanying gesture.

| 1 | here | whole rt arm vertical P0, palm facing away from <br> body, moves up in direction of fingers |  |
| :--- | :--- | :--- | :--- |
| 2 | turn here | TR, arm moved in direction of fingers <br> (maintaining TR position) |  |
| 3 | turn here | attempt to move rt hand TR again (too difficult?) |  |
| 4 |  | switch to lt hand, arm horizontal pointing rt, hand <br> PDN (fingers pointing down) |  |
| 5 | turn here | moves forearm clockwise, hand still PDN (fingers <br> pointing left) |  |
| 6 | but I want it <br> to <br> come | turns arm (awkwardly) so that, hand still in PDN <br> position, fingers point towards body |  |


|  | forward |  |  |
| :--- | :--- | :--- | :--- |

Table 4: T imagines a wall.
The switch (lines 3-4) between use of right and left hands appears to be a response to the physical difficulty of achieving the desired position with the right hand (see Figure 7).


Figure 7: T switches hands.

We consider what remains the same and what is changed with this switch of hand. The switch allows T to maintain the direction in which the fingers are pointing (down). This may be taken to represent the turtle heading within the vertical plane parallel to the screen. However, in switching arms, she changes the relationship between arm and hand from a turn gesture to a pitch gesture. We use turn and pitch within the conventions set up by the teachers/ researchers and the Logo language, not to suggest that T associates her gestures with these terms. On the contrary, she does not appear to attach any significance to the distinction, focusing solely on the position of her hand and the direction in which her fingers are pointing in order to describe the intended turtle movement. While she is to some extent 'playing turtle' with her hand, she is defining the turtle's movements by using position and heading at the corners of her imaginary wall rather than by using turn and distance as required by the Logo language. The use of the turn and pitch gestures is thus not supporting her move into using Logo code and may indeed have made her communication with teachers/researchers less effective.

In considering the difference between the ways in which teachers/researchers and students were using the 'same' gestures, we distinguish between the two notions of imaging and imagining. We define imaging as using gesture to create an image of the construction of the turtle path. The movement of the hand mimics the movement of the turtle: the forearm is held parallel to the current heading of the turtle and the hand is moved to define the next heading. Thus, as in Figure 8, the gesture indicating 'up pitch' is always relative to the current heading of the turtle. In both episodes 1 and 2, the teacher/researcher gestures were imaging the process of construction of the turtle path.


Figure 8: All these gestures indicate 'up pitch'.

In contrast, in episode 3 student T used apparently similar hand movements to construct very different meanings. For her, the relationship between forearm and hand did not appear to have significance, as she was willing to substitute a pitch down gesture with her left hand for a turn right gesture with her right hand. We characterise her use of gesture as imagining, referring to her mental image of the desired outcome of turtle drawing. In this episode, as in several other episodes of student gesture within the data set, the gesture indicates the desired direction of movement in order to draw the desired outcome, rather than indicating the required type of turn. Thus, for example, a movement in the 'up'


Figure 9: 'down pitch' indicates 'go. up'. direction (within the plane of the screen) might be indicated by use of the spoken word "up" and a 'down pitch' gesture (Figure 9).

Teachers and researchers used specialised hand gestures to communicate with students about three-dimensional movement. Students used the 'same' gestures but to communicate different meanings in relation to turtle movement. Whereas the imaging by teachers/researchers mimicked turtle movement in a kind of 'playing turtle' action, student use of gesture to imagine the outcome of the movement seems closer to deixis, pointing in the direction of movement from a viewpoint outside the turtle. Indeed, one student explicitly refused to accept the 'playing turtle' metaphor offered to her by a researcher:

JA if you imagine yourself as a turtle, how are you going to move?
K it is very uncomfortable imagining myself as a turtle ... erm
JA or imagine your hand
K I don't want to be a turtle.
Pointing is a widespread form of representation of position, common in everyday discourse. While it might appear at first sight that students adopted the specialised gestures employed by the teachers/researchers, the students' use and interpretation of these gestures may be closer to the resources of everyday discourse than to those of the MaLT microworld.

While the scope of the teaching experiment described here was limited, our observation of these different ways of gesturing turtle movement leads us to ask whether the 'playing turtle' metaphor is fully adaptable and relevant to the three dimensional context? While we have extensive knowledge of our own body movement in the normal two-dimensional horizontal plane that can be connected to the movement of a turtle in the vertical plane of the computer screen, our experience and knowledge of movement in three dimensions is much more limited. Many of the movements required of a turtle constructing a path in the threedimensional space of MaLT are impossible for the human body within its normal environment. The extra leap of imagination required to 'play turtle' as if in control of an acrobatic aircraft or perhaps in deep water with highly developed underwater manoeuvrability may be too great for genuine body syntonicity.

## The use of the variation tool

## ETL analysis

In task 3 the need to design figures in different planes of the 3d space challenged pupils to move the focus of their attention from directed turns between lines and planes to directed turns between two similar geometrical figures which is related to the conceptualisation of a dihedral angle in 3d space. For example, most groups of pupils recognised that in order to construct windows in two consecutive walls (planes in mathematical terms) the use of the commands 'rr/lr' or 'up/dp' was needed. During this construction process students easily identified the dihedral angle defined by the two consecutive windows (i.e. rectangles) and used the terminology familiar to them from 2d geometry in order to describe it. However, all groups of pupils had difficulties in identifying its measure. For instance, students of group C characterized the dihedral angle drawn by the turtle as an acute and not as a right one as it was the case, although they had commanded the turtle to leftroll 90 before drawing the second window (Figure 10). It seemed that students had focused more on the visual characteristics of the figural representation and were confused by the 'distortion' of the dihedral angle as a result of the use of a vanishing point in the line of horizon of the TS designed to strengthen the sense of depth in the representation.


Figure 10

However, the use of the two new types of turtle turns coupled with pupil's explorations of angle as a dynamic amount that could by dynamically handled and changed sequentially using the functionalities of 1 dVT facilitated further the visualization of different planes in 3d space. For instance, in task 4 (Figure 4) the same students decided to use not a fixed turn measure but a variable after the rightroll command so as to simulate the opening and closing of a door.

The use of 1dVT allowed students to view their constructions from different perspectives which might have minimized the 'distorting' effects of static 3d the need to construct initially four rectangles dragging on the slider :c on the 1dVT. representation and prompted them to
focus on the measure of the turtle's turn in the Logo code. The more the students appeared accustomed to the conventions used in the 3d simulated space the more they were able to coordinate the visual characteristics of the drawn angles with their measure related to turtle's turns. For instance, during the dynamic handling of the revolving door simulation (task 5) students were able to overcome the difficulties faced earlier during task 3 and to recognize the four consecutive right dihedral angles created between the four rectangles around the common vertical side of the four rectangles (see Figure 11). Experimenting with the variables of the procedure 'slide' (see below), which was given ready-made to them (task 5), so as to create a revolving door moving around group C students progressively got able to handle different aspects of angle simultaneously. Since for random values of the variable (:c) four parallelograms appeared around the common side of them, S1 compared the visual outcome with a door and recognized


Figure 11. The revolving door simulation.
S1: Wait, we should move it here first. It's the angle of the rectangle [moves the slider (:c) to the value 90 so as to construct 4 rectangles], so as to become like this (i.e. the door) and then probably turns with this [moves the slider (:e)]. Let's see...

S2: Yes, it definitely turns around with this $[$ i.e. slider (:e)] as it has 1 lr .
S1: Yes, but we don't only want it to turn, we also want it to move even further down.
S2: I should change here [He puts the slider (:d) to the value 90 so as to have the simulation in a vertical position].

S1: Yes, 90 is fine.
S2: Now, with this [points to the slider (:e)] it turns around normally.
The above excerpt accompanied by the respective Logo code indicates that by dynamically manipulating the geometrical construction with the use of the 1dVT students created meanings in relation to angle (i) as a constitutive element of a figure which is defined and stay fixed (variable :c), (ii) as a means to move from the horizontal plane to the vertical one in
relation to the viewing axis of the user which is again defined and stay fixed (variable :d) and (iii) as a means of constantly changing planes in 3d space (variable :e) around the common vertical side of the four rectangles (Figure 6).

## IOE analysis

In Session 4, a number of tasks were posed that involved using the 1 d variation tool. I these tasks, students used the variation tool in order to explore pre-constructed models:
manipulating a one variable model in order to 'close up' a triangular prism - objective to identify the triangular cross section as an equilateral triangle with $60^{\circ}$ angles
manipulating a two variable model to 'close up' a triangular prism - objective to recognise the relationship between an internal and an external angle of the prism
manipulating a four variable model to construct a triangular prism - objective to identify relationships between angles and to identify edges that needed to be equal

Because of the context in which we were conducting the study we were able to make less use of the 2 d variation tool than had been anticipated. A task involving the 2 d variation tool was posed as a follow-up to task b), manipulating both variables simultaneously. Very few students attempted this task and the data related to it is extremely limited. We therefore focus only on the use of the 1 d variation tool.

All the students attempted task a). Many were able to connect the values displayed on the variation tool with the movement of the shapes, in particular connecting the number 60 to the angle of an equilateral triangular face of a prism. This developed the angle measure repertoire of many students in the class. At the beginning of use of MaLT with the class, the 90 degrees was the only angle value used spontaneously by students either in discussion or in Logo commands. Throughout the experimentation, we observed students using 90 as a default angle value, inserting it in Logo commands before analysing the shape they were attempting to construct, then adjusting the value (for example by halving) or repeating the command in order to achieve the desired angle. It appeared that 90 was the only angle value that they were able to use with confidence (if not with accuracy). Before this session, we had not observed students using 60 as an angle value; they had tended to use 45 for any non-right turn, apparently relying on a halving strategy. The use of the variation tool for task a) provided a further angle value whose special significance emerged from its role as the outcome of successful completion of the task. For many of the students this emergence appeared to allow them to construct a chain of signification connecting the symbolic 60 on the variation tool slider to the triangular shape displayed in the turtle screen and hence to previous experience in mathematics lessons, working with angle values in equilateral triangles. In whole class review of this task, the 'special' nature of the 60 degree angle emerged as a new piece of 'common knowledge' among the class.

Some students, however, although they could successfully manipulate the sliders, had difficulty connecting the numerical display to the shape. T had called the teacher (GD) for
help when faced with the task of explaining why the value of 60 on the slider made the triangular prism join up.

1 GD When you look at just the triangle bit, what type of triangle is it?
2 T triangular prism?
3 GD Just the triangle, forget about the fact that it's three dimensions. What type of triangle is it?

4 T ... (hesitates - no answer)
5 GD What can we say about the lengths of the sides?
$6 \quad$ T They're all equilateral
7 GD Exactly. It's an equilateral triangle. So what are the angles in an equilateral triangle?

8 T All the same
9 GD They' re all the same and they have to add up to?
10 T One hundred and eighty
11 GD So what's the size of an angle?
12 T One hundred and eighty
13 GD They add [emphasized] up to one hundred and eighty
14 T forty five degrees
15 GD What's three lots of forty-five?
16 T pardon?
17 GD What's three lots of forty-five?
18 T um ...
19 GD Does it make one hundred and eighty?
20 T ... (shakes head - no)
21 GD So what number do we need for three lots of that same angle to make one hundred and eighty degrees?

22 T ... seventy? ... sixty? (very quiet and hesitant)
23 GD ... So what's the angle in an equilateral triangle? You think about that.
24 T Is it forty-five? (more loudly and confidently)

The teacher (GD) attempts to lead T through a sequence of logical steps to construct the following argument: this is an equilateral triangle; the sum of the angles of any triangle is 180; an equilateral triangle has three equal angles; the sum of these three angles must be 180 ; therefore each of the angles must be 60 . However, $T$ does not join successfully in the IRE sequence to co-construct this argument. The apparent failure of this dialogue to help T make a connection between the value 60 and the size of the angle seems related to the discontinuities in the theme of the discussion at lines $14 / 15$ and $22 / 23$. GD changes the theme from angle to calculation and then back again. T's lack of connection between the two themes is evidenced by her request for clarification "pardon?" at line 16 and by the contrast between her hesitance at line 22 and her confident repetition of her answer forty-five at line 24 . T's difficulty in dealing with thematic discontinuity also seems evident in her lack of any answer after the shift at lines $2 / 3$ between considering the 3 D representation in the MaLT turtle screen and considering an abstract equilateral triangle. It seemed surprising to us, as well as to T's teacher, that the juxtaposition of the symbolic 60 displayed on the variation tool and the visual display of an equilateral triangular prism on the turtle screen did not support T in recognising at least that 60 was likely to be an answer to some of GD's questions. It seems that, in spite of the dynamic relationship between variation tool and turtle screen object and in spite of the naming of the variable 'angle', T was unable to connect the two representations.

The question emerges of why some students were able to make use of the relationship between the symbolic output of the variation tool and the visual feedback in the turtle screen to form a chain of signification that enabled recall and subsequent use of the 'special' role of 60 degree angles in equilateral triangles, while others. like student T , were not able to do so. Clearly, although the class had had similar previous curricular experience, there was some variability within the group of students with respect to the security of their knowledge about angles in triangles. None were fluent in their angle knowledge but, while for some the symbolic-visual connection was sufficient to support them in recalling and possibly strengthening previous experience, others did not have a sufficient basis to be able to make the connections. This observation has implications for the suitability of use of MaLT with students; the representations that MaLT offers necessarily interact with those students are already familiar with. It seems a certain threshold level of knowledge is necessary to allow fruitful chains of signification to be formed.

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## VI. Cruislet cross-case analysis for DEL18

## VI.1. Identification

Teams involved : ETL and DIDIREM

## VI.2. Contextual elements

|  | ETL | DIDIREM |
| :--- | :--- | :--- |
| School level | Grade 10 (two classes) | Exp 1 : Grade 11 (2 classes) - <br> $16 / 17$ ans <br> Exp 2 : grade 9 (2 groups of <br> pupils) -14/15 ans |
| Number of classes and <br> students involved | 2 classes: 24 students <br> Class A1: 12 students in 6 <br> pairs <br> Class A2: 12 students in 6 <br> pairs <br> 2 students in each pc | Exp 1:2 classes : 60 students |
| Number of groups : 18 students <br> hours of the experimentation | Class A1 : 20 hours in total, 2 <br> days/week <br> Class A2: 8 hours in total, 2 <br> days/week | Exp 1:3 sessions (4 hours) |

## ETL Context

Apart from the lessons of informatics -which is taught in the computer lab- the teaching of the various subjects in Greek high schools usually, takes place in a typical classroom where students sit in two at desks looking at the front of the classroom where the blackboard and teacher's desk are placed. Although computer use for doing mathematics is suggested in high school curriculum, mathematics teachers usually don't use computational environments as it isn't officially part of the curriculum. Consequently, students are not familiar in using computer environments to explore mathematical ideas. In addition, the computer laboratories are usually occupied for the teaching of informatics. On the contrary, students are already familiar, from their out-of-school activities, with 3D game environments and representations such as those provided by Google map. The Cruislet software has game-like features such as navigation in 3D space by avatars and geographic maps, thus students are already familiar with the kind of provided representations.

Twenty four students of the 1st grade of upper high school, (aged 15-16 years old) participated in this experiment. Students worked in pairs in the PC lab. Each pair of students worked on the tasks using Cruislet software.

The students were not accustomed in using computers for doing mathematics, but they were familiar with computers and liked using them, as almost the whole class participated in the computer class (available as a course to choose at this school level). On the other hand, concerning the concepts of geographical and spherical coordinates, none of the students had previous knowledge or experience with spherical coordinates and only four of them believed that the acquired experiences during the geography course supported their understanding of the concept of geographical coordinates. Some of the students were familiar with the basic Logo commands (movement of the turtle, such as front, right, etc.) but none of them was experienced in using programming languages. Finally, few students were familiar with map computational environments and especially with Google Earth. Nevertheless, almost all of the students were used to play computer games and most of them were familiar with 3D game environments.

Concerning the mathematical concepts that are embedded in the number of tasks in which students have been engaged, there is a considerable distance from the traditional structure of the mathematics curriculum. In a traditional mathematics class students study the concepts of Cartesian, geographical and spherical coordinate systems within abstract mathematical contexts in a rather static way. They are introduced to the concept of function through static representations provided in their textbooks without having the opportunity to manipulating or change them. Additionally, students are introduced and study the concept of vectors mainly in physical sciences for the description of a number of physical properties such as velocity, force, acceleration.

## DIDIREM context:

In France, comparatively, the use of computer environments is part of the curriculum from junior high school, and since a few years, at the end of junior high school, students validate the knowledge gained in that area through the B2I. The use of specific software especially spreadsheet and dynamic geometry is also part of the mathematics curriculum, but this does not imply a general and substantial use. As in Greece, access to laboratories can be rather difficult and a collective and episodic use of software in the ordinary classroom thanks to a video-projector is more widespread.

DIDIREM developed two different experiments with Cruislet: the first one with grade 11 students, the second one with grade 9 students. For the first one, the experiment was associated with the initial phase of a multidisciplinary project work. The first session was collective with one computer connected to a video-projector, the two other sessions took place in a lab class where students worked by pairs on computers, the teacher having his computer connected to the video-projector. The second experiment took place at the University ParisDiderot. Each year, the mathematics department receives grade 9 students coming from neighboring high schools for a one week stay. In the frame of this program, a 3 h workshop on

Cruislet was offered to the students in parallel with other workshops. The students worked by pairs at computers and their work was supervised by DIDIREM researchers.

The mathematics teachers of the grade 11 students involved in the first experiment regularly used computer facilities both in the classroom and in lab sessions. This was not the case for the grade 9 students involved in the second experiment, even if they had used spreadsheet and Internet at school. Like Greek students, all French students were familiar with computer games.

The fact that the geographic map in Cruislet is the map of Greece made the object not familiar but contributed to the students' motivation and interest. In the design of the experiment, DIDIREM team tried to benefit from this particular situation in the first phase of the pedagogical plan where Cruislet was introduced. In the first experiment, a history-geography teacher was involved and guided the exploration of the map, using places of interest for geographical or historical reasons. He also made the link between the coordinate system of Cruislet and the notions of latitude and longitude introduced and used in geography. In the second experiment, there was no secondary teacher as explained above, but a similar strategy was possible and successfully used as history of Greece and the notions of latitude and longitude are part of the junior high school curriculum.

From a mathematical point of view, the DIDIREM pedagogical plan focused on vectors and trigonometry in 3D space. In France, vectors are introduced in mathematics first in the context of 2D geometry, and this takes place in grade $9^{4}$. Vectors are introduced as objects characterized by their direction, sense and length, and tightly linked with translations (the sum of vectors is associated with the composition of translations). In grade 10, he product of a vector by a scalar is introduced in grade 10 and vectors enter also the scene in physics. The generalization to 3D space comes later (grade 11). Students are not introduced to spherical coordinates in maths in high school. Trigonometry for its part is progressively introduced from grades 8 and 9 (trigonometry in the triangle). In grade 10, angles of vectors and trigonometric functions enter the syllabus. There is no spherical trigonometry in high school. Moreover, none of the students involved in the experiment knew the Logo language.

## The ETL Pedagogical Plan

This plan includes three successive phases briefly described below.

## First phase: Learning to fly (Familiarization)

This phase is considered as introductory to the Cruislet environment and the provided representations. The aim of the tasks that are included is students' familiarisation with:

Geographical coordinates

## Spherical coordinates

Logo programming language

[^4]The 3d terrain scene
In particular, students' are encouraged to explore DDA's functionalities by:
Exploiting geographical or/and spherical coordinates systems for freely navigating upon the 3d map of Greece

Exploiting both systems of reference for displacing the avatar in specific places of the 3d map (e.g. the city of Athens)

Editing basic Logo commands to perform navigation using both systems of reference.
Taking-off and landing the avatar in specific places using either the GUI or the Logo programming language.

Integrating the geographical and the spherical coordinate system in the process of navigation and making conjectures concerning the way that these two systems of reference are possibly related.

## Second phase: Airplanes' chase

In the tasks that are included in this phase students are encouraged to experiment with programs defining the relative displacements of two airplanes by varying the geographical coordinates of their new positions. Reflecting on their actions they are encouraged to explore the rate of change of these positions and formulate the function that defines this dependent relationship. This function is hidden and the students had to guess it based on repeated moves of aeroplane A and observations of the relative positions and moves of planes A and B.

In particular the Logo program copied below is used. The procedures included are a black box to students and students experiment with the result of the execution of these procedures.

## The 'Radar' program

to begin
createavatar("|white| 37.9423 .945000 "|Plane 1|)
createavatar("|red| 37.8923 .925000 "|Plane 2|)
activateavatar("|white|)
setupcamera(15000 0-87-41 0)
end
to radar :a : $\mathrm{b}: \mathrm{c}$
activateavatar("|white|)
setpos(:a :b :c)
wait(1)
activateavatar("|red|)
setpos(:a-0.1 :b-0.05 :c-2500)
print("Coordinates oppos())
if and(and(and(:a>40.73 :a<40.74) and(:b>22.99 :b<23.1)) :c<3001)
[print("Escaped! ) removeavatar("|red|)]
setupcamera(25000 0-87-41 0)
end
Students are actually asked to study the relation between the two aeroplanes, the rate of change of their displacements and consequently find the linear function (decode the rule of the game). In order to decode "the rule of the game", they should give various values to coordinates (Lat, Long, Height) that define the position of the first plane. They are encouraged to communicate their observations about the position of the second plane to each other and form conjectures about the relationship between the positions of the two aeroplanes.

Once the rule of the game is decoded the Logo program is released and the students are encouraged to build their own rules of the game by changing the function of the relative displacements of the two aeroplanes. Initially, they are asked to write down their ideas in order to create the new rules of the game. Then, students interfere into the Logo code and change it according to the rules they decided to create. As a result they come up with new Logo procedures that define the rules of the game they created. Finally, students exchange these new Logo programs and they are challenged to decode the rules of the games that their classmates developed.

## Third phase: The instruments are broken

Students' engagement with the tasks that are included in this phase focuses on the study of the function that defines the dependent relationship between the geographical coordinates of the displacements of an avatar. In particular, they are asked to navigate a specific airplane by setting the geographical coordinates of the new position in a black box Logo procedure. The outcome of this procedure is the displacement of the airplane in a relative position. Students are encouraged to explore the relation between these two positions (the given and the outcome) and actually investigate the existence of the linear function that defines this dependent relationship. Students are experimented by varying the geographical coordinates of the given position and study the respective coordinates of the outcome position. Emphasis is given to the development of interaction between students concerning their observation and their approach to define the hidden functional relationship between the geographical coordinates of the two positions. They are encouraged to write down their observations in worksheets and express them verbally, symbolically and graphically. Finally, students are challenged to verify their conjectures concerning the functional relationship by navigating the airplane towards a specific place on the 3d map of Greece, i.e the city of Rhodos.

## The DIDIREM Pedagogical Plans

We successively present the pedagogical plans corresponding to the two experiments

## First pedagogical plan

The organization splits into several phases
First phase: Presentation of the software
a) Collective presentation of the software using a video-projector. The collective presentation will explore the main characteristics of the DDA:

- virtual exploration of Greece, location of important historical and touristic places by scrolling the 3D map and zooming in/out,
- creation of an avatar, and presentation of the different existing modes for moving this avatar (entering a final position in (lat, long, height), a vector displacement in spherical coordinates or a city name),
- presenting the camera system, and looking for reasonable parameters for it,
- exploiting the interrelation between representations (for instance for getting the coordinates of a particular place),
- exporting the displacement of an avatar into a Logo procedure.
b) Collective programming of a first trip with one stop, for instance a flight from Athens to Sparte. Programming should first be done by using absolute positions, then by using displacements. Angle Fi could be 0 for the first step, then vary to produce a change in altitude.
c) Small group work for preparing variations of the initial trip.
d) Collective discussion: listing the questions raised this first activity and the solutions found or advances reached by the different groups.


## Second phase : Preparing and programming trips

a) Collective discussion: coming back to the questions raised at the first session if necessary.
b) Small group work: each group completes at least one travel.
c) Collective discussion and synthesis : How to prepare a trip? What data are necessary? How to get these? How to program a trip? What has been learnt about the different commands?
d) Small group work on a new problem. Adding a turn around Olympe Mount
e) Collective discussion: Comparing the strategies used. How to design a circular trip at a given altitude? What to change to design an helicoidal trip? Or to design a spiral trip at a given altitude?

## Third phase : Coping with the wind effect (not implemented in the classroom)

Fourth phase : Project work (not implemented)

## Second pedagogical plan:

## First phase

The teacher-researcher uses the video projector to illustrate the first important functionalities of the software:

- the two graphic zones in two or three dimensions
- the functionalities of zooming and scrolling in 3D offered by the mouse
- the information button (I) that allows the display of the region, the longitude, the latitude and altitude of the current point.

Students are invited to familiarize with these first functionalities by looking for longitudes, latitudes, and altitudes of some given cities, then to choose themselves a particular location and characterize it. The first phase ends with a collective discussion in form of report where the values obtained by the different groups are compared and the differences in altitudes explained. The extreme latitudes and longitudes of Greece are also collectively looked for, and the particular locations selected by some groups identified from their coordinates.

## Second phase

The teacher-projector uses the video projector to explain how to introduce a plane (the menu "avatar") and how to move it in two possible ways:
12. Firstly, using the menu "select a destination", or entering into the menu "position" the latitude, longitude and altitude of the selected place. During this explanation, the teacher asks students to introduce a plane on their computer and to bring it to Athens, then to Patras.
25. Secondly, using the menu «direction» and entering a vector for moving the avatar defined by its spherical coordinates (Theta, Fi, Rm). The teacher illustrates this operating mode via some examples and students are invited to do the same. Then, he asked students to anticipate the effect of some selected movements (using angles corresponding to horizontal movements in the four cardinal directions in particular)
At the end of this phase, an impossible movement is introduced in order to have the students face the kind of feedback provided by Cruislet in that case.

Students are then invited to change the form of their avatar into a helicopter and a challenged is proposed to the groups. They have to land the helicopter as close as possible to the summit of on Mount Olympus. The second phase ends again with a collective discussion based on the reports of the different groups. Students explain their strategies and announce the altitude at which they managed to land the helicopter. Different strategies are compared. The question of reducing the number of movements is raised and put in relation with the addition of vectors.

## Third phase

During this first phase, students must organize from the altitude of 400 m above Athens a flight to Sparta with the constraint that it should be as short as possible. Then they have to:
13. identify the two mountains on the journey from Athens to Sparta, their position compared with the cities Athens and Sparta and their altitude;
14. define a strategy for bypassing or flying over two mountains.

If they choose to fly over the two mountains, they are asked to:

- use the map of Greece for identifying the value of Theta corresponding to a flight from Athens to Sparta;
- choose at least two intermediary points for defining the flight over the mountains (distance of Athens, altitude);
- calculate the angles Fi and rays Rm for the different parts of the flight from Athens to Sparta

This third phase is also the object of a collective discussion of the strategies used that are visualized and tested, and the calculations involved are explained to lead to the comparison of journey lengths.

## Fourth phase:

Students are collectively presented with an acrobatic flight corresponding to the following program:

Connecting the visual characteristics of the flight and the program, they are asked to interpret the commands, then to transform the program in order to have the avatar describe:

- an horizontal spiral
- an horizontal circle
- an helix,
then freely create acrobatic flights.


## VI.3. Theoretical frames

## ETL Theoretical frames

We adopted the approach of students' gradual mathematization within game-like activities in problem situations that are experientially relevant to students. Hence, our intention was to involve students in activities through which they would use symbols, make and verify hypotheses in order to solve a particular real problem in a rich collaborative learning environment. Within the framework of instrumental genesis, we particularly focus on instrumentalization, i.e. the ways in which students learn through making changes to the digital artefact at hand. We studied the idea of pedagogical design of artefacts so that students would inevitably poke, tweak and make changes to their functionalities as part of their mathematizations. Consequently, we saw a helpful relevance in studying mathematizations in a constructionist environment as path towards clarifying the idea of instrumentalization by design.

We see these kind of artefacts like Crusilet as designed for mathematizations through instrumentalization and call them 'half-baked microworlds'. Students were provided with tools allowing them to navigate avatars by making choices between spherical and geographical displacement controllers, study Logo programs of sequential functional displacements and finally interfere into the Logo code and change it according to the rules they decided to create. Half-baked microworlds are designed to incorporate an interesting idea but at the same time to invite changes to their functionalities and are mediated to the targeted users as unfinished artefacts which need their input. In that sense, such kind of microworld invites constructionist activity and they are designed for mathematizations through
instrumentalization. Finally, although constructionism is considered to be an individualistic theory of learning, we studied students working in pairs or threes collaboratively including their verbal exchanges and argumentations.

## DIDIREM theoretical frames

The DIDIREM pedagogical plan relies on three different theoretical frames: the Instrumental Approach, the Theory of Didactic Situations, and the Anthropological Theory of Didactics. The impact of these theoretical frameworks on the design is evident when one considers the following points:

- In the design of the two experiments, a specific attention is paid to Cruislet instrumentalization. The first contact with Cruislet is organized in two steps: first focusing on the geographical part of Cruislet without introducing avatars, then introducing avatars and the different ways these can be moved. It carefully alternates collective and individual phases. Programmation in each case is introduced at a later stage. Instrumental needs are also reduced, especially in the second and shorter experiment, by fixing the parameters of the camera. These design choices result from a analysis of Cruislet characteristics guided by the instrumental approach.
- In the a priori analysis of Cruislet, particular importance has been given to the feedbacks provided by Cruislet and the way these can support the autonomous activity of students. The tasks proposed to them have been designed in order to take the maximum benefit of these feedbacks. One can see there an evident influence of TDS.
- The reference to ATD made the DIDIREM team especially sensitive to the existing distance between Cruislet, the French mathematics curriculum, and more globally the kind of software used in mathematics courses, and to the ecological problem resulting from this situation. The team thus tried to find for this DDA a possible "habitat" and "niche" in the French educational system. This led to the choice made of experimenting Cruislet in the frame of pluridisciplinary projects in grade 11, and in university workshops in grade 9. This also influenced the educational goals chosen for the two experiments (see below) and for instance the fact that tasks leading students to revisit old knowledge (angles, vectors, trigonometry) in new and non usual contexts were designed.


## VI.4. Comparison of didactical functionalities

The comparison of didactical functionalities for the two experiments is summarized in the table presented below..

|  | ETL | DIDIREM |
| :---: | :---: | :---: |
| Tool Features | Multiple linked representations | Proximity with out-of-school technology (maps and avatars) <br> Spherical coordinates, angles, vectors, Logo programming making possible access to programming activities and to differential vision of curves |
| Educational Goals | The investigation of the mathematical meanings that students construct regarding the notion of function as co- variation while navigating in 3d large scale spaces. | Use Cruislet characteristics for connecting school mathematics and out-of-school activities, and making students sensitive to the mathematics involved in social technology. <br> Use Cruislet potential for having students work on 3D representations, angles, vectors, linking this work to displacements in 3D space. <br> Reinvest prior knowledge in nonstandard settings (angles and trigonometry). <br> Familiarize with notions of mathematical interest beyond the secondary curriculum (ie spherical coordinates) <br> Introduce students into Logo programming, giving sense to Logo procedures as records of displacements, showing the power of iteration and offering new differential visions on geometrical curves |
| Modalities of employmen t | Students by exploiting the geographical and spherical coordinates systems of reference both in GUI and LOGO editor tab, make hypothesizes and form conjectures concerning the dependent relationship of the displacements of the avatars. Although the students' engagement supported initially by the gamelike characteristic of the activity, it | A selection of tasks organizing a careful progression in Cruislet instrumentalization, trying to maintain a tight connection between the geographical features of Cruislet and avatar moves, proposing challenging problems making sense in out-of-school activities, offering students the possibility of reinvesting their mathematical knowledge in new and unusual contexts, and allowing students |



## VI.5. Results of the cross-case analysis together with illustrative examples

The experiments carried out were thus quite different and not easily comparable. In what follows, we present the parts we have selected in the two-experiments for showing these differences. Then, in the last part, we exploit these and the whole experimental process for drawing lessons for the connected theoretical landscape.

## ETL experiment

The data of ETL experiment consists of audio and screen recordings as well as students' activity sheets and notes. The data was analyzed verbatim in relation to students' interaction with the provided representations. The focus of the analysis was on the process by which implicit mathematical knowledge is constructed during shared student activity. As a result, students through a process of mathematization of geographical space constructed several meanings concerning the concepts of functions, coordinates and vectors. These meanings were categorized in clusters that accordingly rely upon each concept. The most interesting categories of students' meanings regarding each mathematical concept are presented:

## Functions

Students engaged with the notion of function, through their experimentation with the dependent relationship between two airplanes' positions, which was defined by a black - box Logo procedure (Figure 3). In their attempt to find out the hidden function, they were able to coordinate changes in the direction and the amount of change of the dependent variable in tandem with an imagined change of the independent variable. Our results indicate that students developed covariational reasoning abilities, resulting in viewing the function as covariation.

Initially most of the students expressed the covariation of the airplanes' positions using verbal descriptions, such as behind, front, left, etc. as they were visualizing the result of the airplanes' displacements. Students experimented by giving several values to geographical coordinates in Logo and formed conjectures about the correlation between airplanes'
positions. Through their interaction with the available representations, they successfully found the dependent relation of the function in each coordinate, resulting in their coming into contact with the concept of function as a local dependency.

From a theoretical point of view, students' gradually mathematize the game-like activity by instrumentalizing the provided half-baked microworld. In particular, students gradually incorporated mathematizations which were perceived as functional tools to play the game or to solve the task

## Coordinates

Students didn't always choose one system of reference to navigate in space, but several times combined both to make a displacement. In this way they created links either between distributed coordinates (e.g. height of geographical and fi of spherical) or between all three of coordinates for the two systems of reference.

In their attempt to place the plane at a specific height, students used primarily the height coordinate. However, there were some teams that were using spherical coordinates to carry out almost all displacements. Based on students' actions on a team like that, students were trying to find a way to raise the airplane's height to a specific value, while utilizing the spherical coordinates. In fact one of them gave the idea to use the fi coordinate and raise the airplane by asking the other one: 'The height is fi?' and afterwards he edited the fi coordinate's value in order to raise the plane. This statement is interesting as the student endeavour to create meaning around the fi angle that represents airplane's perpendicular angle, in relation to the height that the plane will be placed.

Another episode where students create a link between coordinates is that of longitude and theta coordinates. In the following episode the students of a team argue about the system of reference that displaces the airplane 'right - left'.

S2: It goes right and left. (referring to longitude)
S1: Right and left.
S2: Yes.
S1: No. Theta is right and left.
S2: These are the degrees.
S1: Yes, the degrees it turns to the left or right.
S2: I'm saying to displace at the same time.
This episode is interesting as it depicts the way students verbally express the way they realize the displacement while using longitude or theta angle of spherical coordinates. In both cases they use the expression 'right - left' giving the displacement a sense of direction. However, S2 supports that longitude doesn't have to do only with turning like theta, but with displacing as well. The way he externalizes his thought demonstrates that he is aware of the
interdependent relationship between longitude and theta. Concluding,we could consider that the instrumentalisation process concerning the different coordinate systems was based upon the modalities of use of the available representations built in the DDA.

## Vectors

While interacting with Cruislet environment, students defined the vector of displacement and through this activity they got involved with the notion of vector. As a result, several meanings emerged concerning vectors and their properties.

## Magnitude

Vectors' magnitude is represented by R in spherical coordinates, so it had to be defined when this system of reference was utilised. During their experimentation students realized that R was remaining constant for a displacement between two specific cities and additionally that was independent of the direction of the displacement. In the following episode students displace the airplane between two cities in their attempt to find their distance.

S1: This must be their distance. (Shows the vector created by airplane's displacement from Arta to Amfissa)

S2: Yes. But how can we find it?
S1: The R m. (Meaning $R$ in spherical coordinates).
S2: No, it's not $R$ m.Oh, you're right! Wait. (Displace the airplane from Amfissa to Arta and they watch $R$ values in direction).

S1:You see? It's the same.
The interesting issue is that although they displaced the airplane towards one direction, they wanted to verify that the distance was remaining constant for the inverse displacement as well. If fact S 1 used this as an evidence to persuade S 2 that R represents the distance between the two cities. Our interpretation of S1's way of thinking is that perhaps he used his intuitions or pre-existed knowledge to apply a property of vectors' magnitude in this particular situation

## Addition of vectors



An interesting episode was that of a team that used intuitions to identify the resulting displacement if this is defined by multiple displacements. This was occurred while students were trying to construct the rules of a game for the other team. To be more specific, students' idea included the relative displacement of two airplanes, based on planes' coordinates. Here we focus only on the correlation of two planes' displacement (named red and blue by students), as they were moving relatively to theta angle and particularly their dependence can be represented as Thetablue $=$ Theta white +180 o . One of the preconditions of the game was also that the first (white) must go to a particular city (i.e. Thessaloniki) to end the first phase of the game. Initially students sketched their idea in order to explain it to the teacher, as shown in Figure 1. In the following excerpts, the students explain their drawing:

S2: As we go up, the other, the spy, will go down contrarily, towards Crete. [...] Let's say, if we go 10 step upwards, he goes down 10 step downwards'.

S1: Blue is conversely commensurate. That is to say, we go 10 meters, he goes 10 meters above. When we get to Thessaloniki, he will get to Rethymno.

From their dialogue we can assume that they were thinking about multiple displacements, as specified by the length of each displacement (i.e. 10 meters). We see that S 1 seems to think of the result of these displacements as he mentions the final destination of each airplane. The interesting thing is that he argues that when the first will be at a specific city, the other will be at a specific city as well, independently of the number of displacements, implying that he used his intuition to add the vectors of displacements and find the final destination of the 2nd plane.


As the researcher was not sure if S1 used vectors' addition, she asked him to draw another figure and picture planes' position when the displacements would not be at the same line and asked him if the second airplane would be placed in the same city as in the first case. The student answered 'If we go to Thessaloniki, he'll be at Crete' and draw the schema shown in figure 2 . From his drawing we can see that although he hasn't added the vectors graphically he is thinking that the only thing that matters is the starting and the ending point.

So whatever the direction of vectors would be, the second plane would be placed in a specific city, taking into account that there is a dependent relationship between the two airplane. We find this episode interesting, due to the way students use their intuitions to express
mathematical meanings without using vector's terms, that is to say without mathematical formalism.

As it is clear evidenced, students gradually mathematized the provided half-baked microworld and instrumentalized the provided functionalities by making and verifying conjectures, generalizations and formalizations concerning the notion of vectors.

## DIDIREM (Exp $1 \& \operatorname{Exp} 2)$

For the first experiment, the analysis showed that, in spite of the interest shown by the students for working with the software, instrumentalization of the different representations and the coordination between these required by the piloting of avatars took more time than anticipated. This was evidenced for instance by the distance observed between collective achievements and personal or group achievements, by he limited use of some representational possibilities (3D controller), and by the limitations observed to the a-didactic functioning expected.

The analysis also attracted the attention of DIDIREM researchers on the mathematical requirements of the tasks proposed to students in the first phase of instrumentalization of Cruislet (the risk of cognitive overload was certainly under-estimated in the design of the tasks), and also on the influence of institutional norms on teachers' decisions even if the specific context of TPE was less constrained.

The second experiment contrasted with the first one as main Cruislet features were proved to be quickly accessible to grade 9 students thanks to the changes introduced in the scenario in terms of tasks and of the tight interaction between the group and collective work along the session. Nevertheless, similar difficulties were observed with the design of a flight under constraints requiring the use of some trigonometry and Pythagoras theorem. Of course, the reduced length of the workshop and the limited number of implementations (2) nevertheless obliges to be careful in the generalization of these positive conclusions.

In what follows, we focus the analysis on the part involving Logo programming in the two experiments. This choice is motivated by the fact that the corresponding tasks were directly inspired from tasks proposed by ETL team on the one hand, and also because even if inspired by ETL, the scenarios in which they take place are quite different that those ETL would design. Moreover the two experiments show two different ways of exploiting Logo programming of geometrical flights.

## Experiment 1: a progressive approach of iteration.

## A priori analysis

The plan, discussed with the two teachers, was to make students program flights involving the repetition of the same action. An equilateral triangle was chosen as the simplest repetitive figure. Because at first the students do not know the LOGO structure for repetition, it was expected that they would simply write three lines with the same action (phase 1). Because we thought that an a-didactic phase conducting students to imagine by them the necessity of a repetitive structure would not be possible in the limit of the experiment (less than one hour
was devoted to the whole approach), we chose to give students a Logo program realizing an equilateral triangle by way of this repetitive structure (second phase). In order that students have to analyse the program and understand the structure, we did not choose the same triangle: we chose a triangle in the vertical plane and asked students to adapt the program for a flight in the horizontal plane like they did in the first phase. We expected then that students would understand the structure as interesting means for writing 'economically' programs for repetitive figures.
In order to assess this understanding, we asked them to program a regular hexagon flight (phase 3). We then thought that students would be able to understand a circle as a regular polygon with 'many' sides. Thus the last question (phase 4) was to program a circular flight.

Note that in phase 2, the requested change was related to the 'body of repetition': changing the iterative variable from teta to phi, while in phase 3 and 4, it was asked to change the number of repetitions.

## A posteriori analysis.

Due to the difficulties they encountered with the previous work (trigonometry), few students actually went further than phase 1 (equilateral triangle without repetition), and no one did phase 4 (the circle). Students that tackled the phase 2 task, correctly changed the Logo program. For the hexagon (phase 3) some reused the repetitive structure, while others just wrote six identical lines.
The experiment was clearly too short for a conclusion. We can only think of this plan as a possible approach of repetition in a course about 'algorithmic' that should become part of the French math curriculum. Note that this plan is consistent with a classroom organisation where students work alone or in pair on a computer. In this organisation, the teacher cannot 'explain' the repetitive structure. Students have to make sense of this structure alone.

## Experiment 2 : from acrobatic figures to geometric flight.

## A priori analysis:

In this second experiment, geometric flights are used both for introducing students to Logo programming and for enriching their vision of curves with a local and differential perspective inspired by Papert's view regarding the way Logo could renew the perspective on such geometrical objects.

In contrast with Expl, this work was made collectively by a small group of students under the direction of a teacher. In addition, it was not a regular course.

Due to the constraints of time of the experiment and the absence of familiarity of students with programming, the choice has been made to propose a first Logo program to students and ask them to make sense of it by executing it and connecting the trajectory of the avatar with the Logo program.

A general aim was to introduce students to "the power of computing" and then the choice for this part of the experiment was to start from a 'surprising' Logo program, that is to say that
the program is short, but the resulting flight is complex. This program had been proposed by the Greek team in the familiar scenario. There is an iteration on the three parameters defining the displacement: theta, fi, and r.


Figure 1: Acrobatic flight
Make "theta 0 make "fi 0 make "r 0 make "i 1 repeat 720 [ SETDIR(:theta+:i :fi+:i :r+:i/2) make "i :i+1 wait(1) ] camdist(50000)

Once students have understood the program, they are asked to modify it in order to produce an horizontal spiral flight, and then a circular flight.

## A posteriori analysis

After executing several times the program, the students decoded it by themselves and changed it for a spiral (as shown in figure 2 below).

Make "theta 0 make "fi 0 make "r 0 make "i 1 repeat 720 [ SETDIR(:theta+:i :fi :r+:i/2) make "i :i+1 wait(1) ] camdist(50000)

To program a circle, there was a difficulty, because they first erased $+: i / 2$ in $: r+: i / 2$, like they did before for :fi. They first did not pay attention to the initialisation. The avatar did not move. Students were surprised and tried to understand why. There was a discussion about the


Figure 2: The spiral flight
meaning of $: r$ and its value in the program. In the discussion two positions appeared some students thinking of :r as the circle's radius and others correctly interpreting :r.

They tried increasing values or :r until they got a sufficiently big circle with the following programm:

Make "theta 0 make "fi 0 make "r 100 make "i 1 repeat 720 [ SETDIR(:theta+:i :fi :r) make "i :i+1 wait(1) ] camdist(50000)

They also noticed that the avatar draws twice the circle and made the connection with the number of iterations (repeat 720). They noticed the increasing relationship between $: r$ and the radius, but there was no time to better specify this relationship.

As a summary of Exp 1 and 2, we can say that they represent two different approaches, based on the modification of an existing repetitive program. Working to understand and modify the program was perhaps more motivating in phase 2 , because the resulting trajectory is amazing. This task was thought feasible thanks to the collective organisation of the work.

## VI.6. Potential offered for the theoretical landscape

The cross-case analysis of the two Cruislet experiments is quite interesting for several reasons. Among the different DDAs developed and used in ReMath, Cruislet is without any doubt the one which is the most distant from the software usually used in mathematics education, and its representations the most distant from those used in secondary mathematics education. It is also the DDA which appears the closest to out-of-school widespread technology. Its geographic maps evoke systems such as Google earth or the IGN maps in France, the moving of avatars can evoke a variety of digital games beyond the software devoted to airplane navigation. To this adds the fact that DIDIREM, the alien team for this artifact has developed a DDA: Cassyopée very far from Cruislet, refers to theoretical frames quite different from those used by ETL, and experiments in a context that has proved to be especially constrained from an institutional point of view. Developing a comparative analysis
of the two designs and of their outcomes, making sense of the differences and similarities observed appears thus as an especially challenging but potentially insightful perspective.

At the level of similarities, there is no doubt that the two teams are sensitive to the specific characteristics of Cruislet mentioned above, an especially its proximity with out-of-school technology. They consider it important to define an educational goal taking this characteristic into account. This is visible in the tasks designed by the DIDIREM team, even if there are evident differences between the two experimentations. An explicit link is made during the exploration of Cruislet with the history of Greece and with touristic sites. Priority is given to open and realistic tasks from an out-of-school perspective (planning trips under constraints for avatars), but in the second experiment tasks are more presented in the form of challenges and games. Certainly the specific context of this second experiment (a university workshop where mathematicians have first to show that mathematics can be much more attractive and diverse that what students experience in schools) contributes to it. This is also visible in the ETL design of tasks. The tasks are based on the idea of the "Guess my function" game, in order to provoke children to discuss, compare and experiment with dependence relations such as linear functions. Emphasis has been given to build game play activities involving navigation within the 3d representational space giving distance from the traditional structure of the mathematics curriculum. The intention was to involve students in activities through which they would use symbols, make and verify hypotheses in order to solve a particular real problem in a rich learning environment. In that sense mathematics is put to use to resolve game like tasks.

The characteristic features of Cruislet and the distance it presents with most DDAs including those experimented in TELMA also makes anticipation of students behavior and possible cognitive outcomes more difficult. It is interesting to notice that this difficulty does not affect the two teams in the same way, and that this difference can be linked to the theoretical approaches they respectively rely on. ETL team seems quite at ease with such a situation, and coherently with the constructionist perspective it relies on, does not fix precise mathematical goals but investigates what spontaneously emerges from the students' interaction with the DDA and tries to make sense of it. For the DIDIREM team, the situation is not exactly the same. This group has developed a vision of design inspired by TDS and didactical engineering. In such a perspective, the anticipations made in the phase of analysis a priori play a major role in design. The design tries to control and optimize the characteristics of the interaction between the students and the milieu, including here the DDA, through a careful choice of the didactic variables of the tasks proposed to the students and their management. The design also tries to anticipate what can be an optimal sharing of mathematical responsibility between the students and the teacher, and what didactic decisions can help maintain this optimal situation if difficulties appear. Such anticipations were difficult for the first experiment all the more as researchers and teachers were just discovering the version of Cruislet to be experimented, its potential and limitations. The difficulties met in the first experiment, the permanent interventions of the teachers in order to restore the expected dynamics for the classroom situations, the feeling of dissatisfaction that this first experiment generated can be partly explained by these differences in the conception of design. It is worth noticing that the experience gained in the first experiment made the context different for the
second experiment. The sharing of responsibilities was more realistically anticipated leading to a design where collective discussions, collective and group solving of tasks was distributed in a more sophisticated way, and where the tasks themselves were more accessible.

Another interesting point from a theoretical point of view is the way the institutional sensitivity impacted the design of the DIDIREM team. As expressed above, this sensitivity can be expressed in ecological terms. In order to make the use of Cruislet possible in a realistic context, a habitat and a niche had to be found. The distance with the mathematics curriculum made the ordinary classroom context an impossible habitat in the French highly constrained system. But the philosophy underlying the system of projects called TPE (associating mathematics and other school disciplines knowledge for approaching issues of more general interest with some freedom with respect to syllabus constraints) seemed compatible with Cruislet spirit. Thus the DIDIREM choice. The three sessions organized had to familiarize the students with Cruislet in order to make them able to develop a project around this DDA. They had also to attract some of the students to the idea of choosing Cruislet as a support for their project. These conditions explain the selected educational goals and modalities of use. This institutional sensitivity also explains the intertwining in the design between out-of-school ingredients, open and exploratory tasks, and the reinvestment of specific notions such as vectors and trigonometry. Once more the comparison with ETL is interesting. The description made by the ETL researchers of the current functioning of the educational system and of their position shows that their concern is not to find a habitat and a niche for Cruislet in the system but to use Cruislet as a tool for questioning or even suggesting changes to the system. Once more this can be related to their theoretical perspectives. From a constructionist's point of view, the functionalities of the new digital media such as Cruislet provide a challenging learning context where the different mathematical concepts and mathematical abilities are embedded and interconnected. The role of the teacher becomes crucial in designing mathematical tasks where students' enactive explorations will reveal these mathematical notions and put them under negotiation. In the case of Cruislet, navigational mathematics becomes the core of the mathematical concepts that involves the geographical and spherical coordinate system interconnected with the concept of function and the visualization ability.

But these theoretical perspectives themselves cannot be considered independently from the context. As pointed out above, the French system for instance seems much more constraining than the Greek system where a culture of continual search for reform is prevalent.

Another element interesting is the differences in the way the two teams analyze the learning potential of Cruislet, and select the mathematical objects and representations they will focus on. For DIDIREM researchers, vectors and angles are the main mathematical objects. This is not surprising as angles and vectors are the most evident mathematical objects in Cruislet. For DIDIREM, angles in Cruislet offer interesting opportunities enriching those usually offered to students. Angles are immerged in 3D geometry and linked to spherical coordinates. Conversely this situation adds complexity to their manipulation all the more as French students are only used to manipulate angles in 2D geometry. The use of trigonometry for preparing flights under constraints for instance needs to situate angles in adequate planes,
which was very difficult for most of them. Regarding vectors, the first analysis made by DIDIREM researchers of their implementation in Cruislet pointed out the limitations of this implementation. Vectors could be only added and multiplied by integers, thus the vectorial structure was only partially accessible. This made impossible to propose students some quite realistic tasks that had been a priori envisaged such as those where wind effects affected the trajectory of an avatar. Another mathematical potential entered the scene later on due to exchanges with the ETL team. This team proposed a program for an acrobatic flight and this opened new perspectives to DIDIREM researchers: using Logo programming for enriching the students' vision of curves with a differential perspective based on the properties of curvature. This went clearly beyond the school curriculum but seemed easily accessible. Due to the lack of familiarity of French students with programming activities, it was decided to propose them first a program they would have to execute and interpret, and then ask them to change the program in order to generate different curves. In fact, in the first experiment the initial activity was transformed by the teachers into a program associated with a triangular trajectory in a horizontal plane, that the students had then to adapt into a program for a triangular trajectory into a vertical plane, and the accent was put on the understanding of the iteration process. The differential perspective was reintroduced in the second experiment and the results obtained evidenced its accessibility to young students.

The choices made by ETL are quite different as the focus is on the functional relationships between two airplanes' relative displacements. ETL researchers consider navigation as a dynamic function event. The function's independent variable is the geographical coordinates of the position of the first aeroplane, which students are asked to navigate, while the dependent variable is the geographical coordinates of the position of the second aeroplane. ETL team consider that the exploitation of the provided linked representations (spherical and geographical coordinates), as well as the functionalities of navigating in real 3d large scale spaces could enable students to explore and build mathematical meanings of the concept of function within a meaningful context. Function is seen as systematic mathematical covariation between specific coordinate values ( $\mathrm{x}, \mathrm{y}, \mathrm{z}$ displacements or changes to r,phi,theta values). The realistic nature of the domain of these functions adds to the potential for students to generate meanings pertaining to domain and to functions which may provide a realistic game (e.g. it would be difficult to follow flights connected with quadratic functions).


[^0]:    ${ }^{1}$ The tree representation has been implemented in the new module developed within the ReMath project.

[^1]:    ${ }^{2}$ The feedback can be provided either permanently, or on demand, or can be limited to two or four verifications during the solving process.

[^2]:    ${ }^{3}$ We only analyze techniques available in Algebraic Line and Cartesian Plane component. Techniques based on algebraic transformations of expressions are also available in Alnuset, namely in Algebraic Manipulator component, which was not used in the experiment.

[^3]:    "Topaze wants the student to succeed; after all, part of the didactic contract is the obligation, for the teacher, to do all he or she can to help the student succeed. But the way he is going about it, is not leading to the student's learning, but to the student's producing a correct answer in spite of not having learned anything." (Sierpinska's notes)

[^4]:    ${ }^{4}$ The syllabus mentionne dis that in place when the experiment took place.

