Information Society Technologies (IST)

Programme

Information Society Technologies

Project Number: IST4-26751
ReMath

Representing Mathematics with Digital Media

Deliverable 18: Integrated Theoretical Framework Version C

Deliverable Number:	18
Due date of the report:	31/05/2009
Actual submission date:	06/07/2009
Start Date of Project:	01/12/2005
Duration:	42 months
Version:	A
Work Package:	WP1
Lead Partner:	DIDIREM
Contributing Partners:	UNISI, UJF, ITD-CNR, ETL/NKUA, TALENT, IOE-LKL Michèle Artigue
Authors:	E. Alexopoulou, J. Alshwaikh, M. Artigue, C. Cazes, H. Chaachoua, G. Chiappini, K. Kahn, C. Kynigos, J.-B. Lagrange, M. Latsi, L. Maffei, M. Maracci, M. A. Mariotti, C. Markopoulos, C. Morgan, F. Moustaki, J. F. Nicaud, B. Pedemonte, G. Psycharis, E. Robotti, C. Sabena, J. Trgalova, F. Vandebrouck, N. Winters
Contact:	Michèle Artigue, artigue@math.jussieu

Project co-funded by the European Commission within the Sixth Framework Programme (2002-2006)

Dissemination Level		
PU	Public	√
PP	Restricted to other programme participants (including the Commission Services)	
RE	Restricted to a group specified by the consortium (including the Commission Services)	
CO	Confidential, only for members of the consortium (including the Commission Services)	

Project Coordinator:

Research Academic Computer Technology Institute (CTI)

eLearning Sector

“D. Maritsas” Building, N.Kazantzaki str.,

University Campus, GR 265 00, Rio, PATRAS

Project Coordinator	Administration Manager
Chronis Kynigos	Argyrakopoulou Roula
kynigos@cti.gr	Argyrako@cti.gr

Partners:

1. Université PARIS 7 DENIS DIDEROT, 2 Place Jussieu, 72251, Paris, France,
2. Université JOSEPH FOURIER GRENOBLE 1, Avenue Centrale 621, Domaine Universitaire, 38041, Grenoble, France,
3. Consiglio Nazionale Delle Richerche, Piazzale Aldo Moro 7, 00185, Roma, Italy,
4. Università degli Studi di Siena, Banchi di Sotto 55, 53100, Siena, Iitaly,
5. National & Kapodistrian University of Athens, Educational Technology Lab, Faculty of Philosophy, Pedagogy and Psychology Secretariat, School of Philosophy (3rd floor), Panepistimioupoli, 157 84 Ilissia, Athens, Greece,
6. Talent Anonimos Etaireia Pleroforikis, Plateia Karitsi 4, 10561,
7. Institute of Education, University of London, 20 Bedford Way, WC1HOAL, London, United Kingdom

Table of Contents

I. Introduction

At the origin of ReMath lies the contrast between the huge efforts performed in most European countries to improve mathematics education through the development and use of Digital Dynamic Artefacts, considering the new means provided by these for representing and manipulating the abstract entities that the mathematical objects and processes are, and the limited impact of these efforts on the reality of school practices. The hypothesis made by the ReMath partners is that the fragmented character of the theoretical frames which have been developed in order to approach learning and teaching processes in such environments on the one hand, the insufficient attention that research in that area tends to pay to contextual issues on the other hand, play an important role in the permanence of such an unsatisfactory situation. Thus the ReMath project of looking for integrative perspectives in terms of theoretical frameworks, and the choice made to closely link their construction to the development of some specific DDAs and to experimentations of these carried out in realistic educational contexts.

What are the outcomes of this project in terms of theoretical integration, in terms of understanding of the role played by contextual characteristics? There are the issues this deliverable focuses on. For allowing an autonomous reading, we have structured this deliverable in the following way. We first review the different phases of ReMath work regarding theoretical integration, and situate this work with respect to the current state of the international reflection in that area. We then try to draw the lessons on the one hand of the DDA development using for that purpose the notion of epistemological profile, and on the other hand of the cross-experimentation through the process of cross-case analysis. Finally, in the last part of the deliverable we synthesize the whole reflection in terms of shared theoretical frame (STF) and connected theoretical landscape (CTL).

II. The different phases of ReMath work regarding theoretical integration

In this part of the deliverable, we synthetically review the different phases of ReMath work concerning theoretical integration, coming back to the deliverables D1 and D9, which constituted the milestones for WP1 along the ReMath trajectory. We pay particular attention to the interaction between theoretical reflection and experimental work, the development of an Integrative Theoretical Framework (ITF) being conceived in ReMath as a cyclic process where these two dimensions must strongly interact.

II.1. The first deliverable

As expressed in the first ReMath deliverable written after six months of existence of the project:

> "Looking for integrative perspectives raises some fundamental questions. What kind of integration can reasonably be aimed at? Does it make sense to look for a unified perspective, an overarching theory or meta-theory encompassing the different existing frames? Or is such a perspective unreasonable, due to the incommensurability of most of the existing theoretical frames? What can only make sense would be then to look for structures and languages in order to better understand

> the characteristics of the corresponding approaches, to organize the communication between these, and to benefit from their respective affordances. If so, can we build such structures and languages, and how can we make these operational?"

These were the questions at the core of the theoretical reflection in the first phase of ReMath. For answering these, the ReMath partners reviewed the work they had developed in the ERT TELMA from the NoE Kaleidoscope, complementing this review by the analysis of different complementary sources coming both from mathematics education research and computer sciences.

Reviewing TELMA work

TELMA work and outcomes influence in a decisive way this first phase of ReMath. It orientates the vision of ReMath partners towards a vision of integration in terms of networking between theoretical perspectives, and makes clear the necessity of investigating the exact role played by theoretical frameworks both in design and use of ICT tools, for understanding and answering networking needs. The notion of *didactical functionality* and the language of *concerns* introduced in TELMA (TELMA ERT, 2006) structure the first version of the ITF. Let us recall the definition of the notion of didactical functionality, introduced for supporting the instrumental vision of the theoretical reflection aimed at:

> "Given an ICT tool, it is possible to identify its didactical functionalities: with didactical functionalities we mean these properties (or characteristics) of a given ICT, and their modalities of employment, which may favour or enhance teaching/learning processes according to a specific educational aim".
>
> "The three key elements of the definition of the didactical functionality of an ICT tool are then:
>
> 1. a set of features / characteristics of the tool
> 2. an educational aim
> 3. modalities of employing the tool in a teaching/learning process referred to the chosen educational aim."

The language of concerns, for itself, can be seen as a consequence of the hypothesis made that the different TELMA teams, even if they live in different contexts and cultures and rely on different theoretical frameworks, more or less face similar problems and ambitions, and have similar sensitivities. The notion of concern tries to capture these commonalities taken as a possible basis for communication and integration of perspectives. It is also supposed that not all of these concerns are considered or given the same emphasis, and even when they seem to be given a similar importance, they are not necessarily expressed, dealt with in the same way, with the same conceptual tools, and that the decisions taken can thus diverge. It is through the identification of the respective attention given to these different concerns, and the precise ways they are approached that TELMA teams try to elucidate the role played both explicitly and implicitly by theoretical frameworks, to identify potential interesting connections and complementarities, and also divergences, potential misunderstandings and conflicts one needs to be aware of when engaging in any kind of integrative work.

From an experimental point of view, TELMA work evidences the interest of a methodology of cross-experimentation where teams are asked to experiment a DDA which has been developed in another educational context, under other theoretical perspectives. Finally, it makes ReMath partners sensitive to the following points:

- "the fact that theoretical frames, be these local or more global, do not fully determine the design of situations aiming at an efficient use of ICT tools. Many decisions taken in the design of such situations as well as in their management in classrooms, once they have been designed, engage other forms of rationality or are shaped by cultural and institutional habits and constraints.

- the fact that theoretical frames themselves often act as implicit and naturalized theories, more in terms of general underlying principles than of operational constructs."

Nevertheless, as also explained in deliverable D1:

> "Nevertheless, what has been achieved in TELMA, however useful it could be, does not give us immediate answers to the different problems we have to solve in ReMath. It tends to show that the metaphor of *networking* is, as regards the idea of integrative perspective, better adapted than the metaphor of *unification*, but it only suggests some hints as regards the strategies we could engage for making this networking productive. Moreover the notions of representation and above all of context have to be more operationally developed. To these limitations adds the fact that the TELMA reflection has been more focused on the design and analysis of uses of ILEs than on their design. In ReMath, design of ILEs is an important component of the project, and the extensions proposed have to increase the potential of communication of the different tools. Even if we are aware that theoretical frames do not determine design, we have to better integrate the work on theoretical frames and the work on design more fully."

Analysing complementary sources

We cannot enter into the details of the results of the analysis of the complementary sources. Their analysis shows evident points of convergence with TELMA views. Theoretical fragmentation is acknowledged and identified as a source of difficulties whose solution cannot reside in building some over-arching theoretical frame able to encompass in a coherent whole a sufficient multiplicity of existing perspectives. The incommensurability of the underlying paradigms is seen to make this enterprise a dead-end one. Beyond, such a philosophical convergence, these complementary sources show the development of different multidimensional grids for trying to make sense of the theoretical diversity of the field. But these tools remain at a level of very general categories and criteria, and the opinion of ReMath partners is that none of them fulfil ReMath goals, all the more as the issues of representation and context on which ReMath focuses, even when mentioned are not deeply developed. Another interesting outcome of the analysis of the computer sciences literature is the evidence of the increase in theoretical complexity when technological design is taken into account and the fact that even if design and utilisation are not independent, an integrative framework taking into account the two perspectives is something specific that cannot result from the mere juxtaposition of two separate tools.

The analysis of the literature provided in deliverable D1 ends by a review of the specific work on representations carried out by the PME Working Group on Representations, which led to two special issues of *The Journal of Mathematical Behaviour* published in 1998 (Volume 17, issues 1 and 2). As expressed in D1, the visions offered are strongly oriented towards a cognitivist view of representations, based upon the dualisms of internal-external and representation-represented. Emphasis is put on the notion of homomorphism and isomorphism seen as structure-preserving mappings between representations (internal or external). Operating successfully with multiple representations, by constructing or using isomorphic relationships or by translating between different representational systems, is seen as an indicator of mathematical understanding, and the role potentially played by digital technologies for developing such abilities is stressed. The image conveyed by these special issues is mainly that of an objectivist view of representations, seen to carry mathematical meanings which have independent objective existence, what leads to judge representations according to the transparency with which they carry the intended meanings. As pointed out in D1, this vision contrasts with the positions supported by socio-cultural theories of learning, linguistics, semiotics and discourse theory increasingly influential in mathematics education, where representation is conceived as a relationship between an object, an individual and (activity, including symbolic activity, within) the social world. Whatever be the diversity of their theoretical positions, ReMath partners are more in line with these last ones.

This study and reflection leads to the first version of the ITF that we present below.

The first version of the Integrative Theoretical Frame

As explained in deliverable D1, the first version of the ITF:

> "is neither a theory more, nor a meta-structure integrating the seven main theoretical frames used in ReMath into a unified whole. It is more a meta-language allowing the communication between these, a better understanding of the specific coherence underlying each theoretical framework, pointing out overlapping or complementary interests as well as possible conflicts, connecting constructs which, in different frameworks are asked to play similar or close roles or functions."

Moreover, due to the process envisaged, the first ITF must be something open and flexible enough to favour its evolution through the planned cyclic iteration. Finally, as it is planned to make sense and become an efficient tool for a wide community of researchers, designers and teachers, the structure and the language have to avoid as much as possible too technical or connoted terms.

As mentioned in the review of TELMA work, the ITF is indeed structured around the notion of *didactical functionality*, and uses the language of *concerns* introduced in TELMA. Moreover, considering the specific focus of ReMath on representations and contexts, the ITF organizes the presentation of concerns around these two focuses for each of the three dimensions attached to the notion of didactical functionality. In order to limit the complexity of the tool, the analysis of representations is restricted to that of external representations and these are considered according to two dimensions:

- representation of objects;
- representation of interaction.

For similar reasons, the analysis of context distinguishes only two categories, that of:

- a local or situational context;
- a global or institutional and cultural context.

It is hypothesized that these distinctions are sufficient, at least in the first stage of the research work, for determining the role played by the theoretical frames in the design and the analysis of the design, considering that these only partially determine the design.

Another important point is that the ITF distinguishes between two main ways of using theoretical frameworks: on the one hand, providing metaphors, general principles and backgrounds, on the other hand providing operational constructs and tools.

The ITF is thus structured into three main parts. The first part deals with the global contextual characteristics of the project under study, which can deal with the design of a DDA or the extension of a given DDA, as will often be the case in ReMath development, with the design of use for a DDA or a set of DDAs, or with the analysis of uses of DDAs. The second part deals with design and is structured around the expression of didactical functionalities with a specific focus on representations and contexts. The third part is concerned with the role played by theoretical frameworks in the effective analysis of uses. Note that, in the articulation of the ITF at that stage, the terminology used is not that of DDA but that of ILE: Interactive Learning Environment, used in TELMA together with that of ICT tool.

Integrative Theoretical Framework
Part 1 : Contextual characteristics of the project under study <u>How are the following dimensions of context taken into consideration at a theoretical level in the project? What constructs are used for this purpose ?</u> - The situational context of the project - The institutional/cultural context of the project
Part 2 : Didactical functionalities and design For each dimension of didactical functionalities, a list of concerns is given. You are asked to grade them from 0 to 5, this grade reflects the level of priority given in design (0 not considered, 5 high priority). In a second phase, you are asked to say what are the theoretical frames you use, if any, when taking into account these concerns, and how you use these. Both representations and contexts are considered.
a) Characteristics of the ILE (or of the set of ILEs if several ILEs are concerned by design) <u>Are the following concerns given a high priority in your design (grade from 0 to 5: 0 not considered, 5 high priority):</u> - concerns about the ways mathematical objects and their interaction are represented?

- concerns about the ways didactic interactions are represented?

- concerns about the ways representations can be acted on?

- concerns about possible interactions, connections with other semiotic systems, including the representations provided by other DDAs?

- concerns about the relationships with institutional or cultural systems of representation?

- concerns about the rigidity/evolutive characteristics of representations?

<u>For those considered, what are the theoretical frames and constructs, if any, which you refer to:</u>

- at the level of general principles and metaphors?

- at an operational level?

b) Educational goals

<u>When thinking about educational goals to be associated to the ILE or set of ILEs, in the design phase, what concerns are given a high priority (grade from 0 to 5):</u>

- epistemological concerns?

- semiotic concerns?

- cognitive concerns?

- social concerns?

- cultural and institutional concerns?

<u>Up to what point are those considered linked to representational characteristics of the ILE or set of ILEs (grade from 0 to 5: 0 no link, 5 strong link) ?</u>

<u>For those linked, what are the theoretical frames and constructs, if any, used for this linkage:</u>

- at the level of general principles and metaphors?

- at an operational level?

<u>Up to what point do contextual concerns shape the vision of educational goals here (grade from 0 to 5: 0 does not shape, 5 strongly shapes):</u>

- local concerns?

- global concerns?

<u>What are the theoretical frames and constructs, if any, used:</u>

- at the level of general principles and metaphors?

- at an operational level?

c) Modalities of use[1]

When thinking about possible modalities of use in the design of this ILE or set of ILEs, what concerns were given a high priority (grade from 0 to 5):

- concerns about the mathematical tasks and their temporal organization ?
- concerns about the functions to be given to the artefact and their possible evolution ?
- concerns about semiotic issues?
- concerns about instrumentation processes?
- concerns about social organization and interactions?
- institutional and cultural concerns?

Up to what point are those considered linked to representational characteristics of the ILE (grade from 0 to 5) ?

For those linked, what are the theoretical frames and constructs, if any, used for this linkage

- at the level of general principles and metaphors?

- at an operational level?

Up to what point do contextual concerns shape the vision of modalities of use (grade from 0 to 5):

- local concerns?

- global concerns?

What are the theoretical frames and constructs, if any, used:

- at the level of general principles and metaphors?

- at an operational level?

Part 3 : Analysis of use

Collection of data

How are concerns about representations and contexts taken into account in the collection of data as regards the use of ILEs?

What are the theoretical frames and constructs, if any, used for this:

- at the level of general principles and metaphors?

[1] Note that we use here a slightly different terminology that the one used in the initial definition of didactical functionalities: “modalities of employment”, but the two of them have to be considered as synonymous.

- at an operational level?
Analysis of data <u>How are concerns about representations and contexts taken into account in the analysis of data as regards the use of ILEs?</u> <u>What are the theoretical frames and constructs, if any, used for this:</u> - at the level of general principles and metaphors? - at an operational level?

Table 1: Structure for the Integrative Theoretical Frame

In the deliverable D1, radar charts and tables are associated to this ITF in order to provide information in a format appropriate for looking at similarities and differences. For instance, radar charts can be used for visualizing the respective priority given to the different concerns. Tables can be used for synthesizing the information given about the theoretical frames and constructs involved. We give some examples of these below.

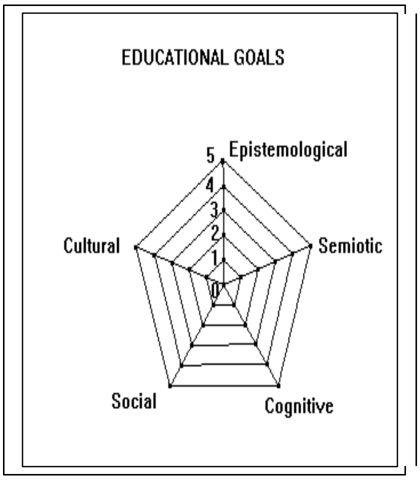

Figure 1 : Example of a radar chart associated to the ITF

Concerns regarding:	Grade	Main theoretical frames referred to	Principles	Operational constructs
Representation of mathematics objects				
Representation of didactic interaction				
Possible actions on representations				

Connections between representations				
Relationships with cultural representations				
Possible evolution of representations				

Context dimension	Grade	Main theoretical frames referred to	Principles	Operational constructs
Local/Situational				
Global/Cultural				

Table 2: Tables associated to the ITF

As shown by this structure, this first version of the ITF does not present a network of theoretical frameworks or concepts but more a structured tool for systematically investigating the exact role played by theoretical frames in the identification of didactic functionalities and associated design, and for making visible the needs the theoretical choices respond to. This version of the ITF is a methodological tool for organizing the connection with the other dimensions of the ReMath project, and especially the cross-experimentation to be designed at this stage. What is presented in the deliverable D1 corresponds thus to a preparatory phase regarding the integrative activity aimed at.

II.2. The second deliverable

The connection with the other WPs, and especially WP2 corresponding to the DDA development and WP4 corresponding to the cross-experimentation was essential in the second phase of ReMath theoretical work as evidenced by the second deliverable corresponding to WP1, deliverable D9 produced in month 18. We synthesize its main outcomes below.

Connection with DDA development

During the first six months of the project, the work on WP2 had been devoted to the definition of precise specifications for the planned extension or development of the different DDAs. It was first considered that this work and the associated deliverable D4 would provide a good basis for connecting WP1 and WP2, and especially for questioning the adequacy of the first version of the ITF for approaching the role of theoretical frames in DDA development. Nevertheless, it quickly became clear on the one hand that the deliverable D4 was too much focused on technical issues and descriptive for being the main support of such an analysis, and on the other hand that we had to look at the history of each DDA to make clear the rationale behind the planned extensions, and become able to establish connections between technical decisions and theoretical approaches. A specific methodology was thus designed, tested with the DDA Casyopée and then generalized to the different DDAs. This methodology included three successive steps:

- Analysis of the information provided in the WP2 Deliverable and articulation of a set of questions arising from it;
- Analysis of the information provided about the role played by theoretical frames by a review of the history of the project;
- Interview with the developers focusing on some particular object involved in the extension work and considered by the designers significant of the way they had worked, and on what steered them to the decisions they had made.

This methodology resulted very productive. The interviews confirmed that decisions taken in the development or extension of a DDA could not be understood without situating it inside a more global picture. As expressed in the conclusion of D9:

> "When an extension is at stake, decisions cannot be understood without considering the history of the DDA itself. The coherence leading the development of the geometric functions in Casyopée or of the tree structure in Aplusix is part of a more global coherence, decisions regarding the introduction of avatars and vectors in MaLT and Cruislet can be traced in the half-baked microworlds previously developed by ETL (Kynigos, 2007 a). The new DDAs themselves are also the product of a history. As pointed out in the interview reports, Alnuset would not exist as it is without Ari-Lab and its numerical line, and the current characteristics of graphical representations in Mopix have progressively emerged, through adaptations and changes, from familiar designs."

As reported in the deliverable D9, the interviews also made clear the influence on DDA design of the educational culture of the different teams, and the fact that very often theoretical frames have acted more as part of this educational culture, in form of thinking habits, choices that seemed to impose because they were considered the natural or the appropriate ones, than in a conscious and controlled way through operational and specific constructs. The authors see a confirmation of this phenomenon in the fact that the transcriptions of the interviews show little explicit references to theoretical frames belonging to computer science or ILE communities. According to them:

> "once the main choices of development have been fixed, framed by educational views and also compatible with the strict constraints imposed to this project, the development itself appears more as a succession of problems to solve where accumulated experience, habits and serendipity play an essential role."

As mentioned above, the analysis of DDA development was perceived as a way of questioning the following choice made in the first version of the ITF: trying to capture the relationships of two categories of design with theory (design of DDA and design of use of such DDAs) in a single structure through the unifying notions of didactical functionality and concern. In the conclusion of the deliverable D9 partly reproduced below, a positive answer is given to this question:

> "Building the ITF and having in mind both the design of tool and the design of uses, we had decided to create only one structure, hypothesizing that through the notion of didactical functionality and the language of concerns, we could capture, even with different balances, the two categories of design between the three dimensions of didactical functionalities. The results from the interviews tend to confirm the pertinence of this choice. Even if the technical problems to be solved in the design of a tool are not similar to those to be solved in the design of modalities of use for a given tool or a set of tools, the interviews show interesting proximities in the ways these problems

> are approached, and in the role that theoretical frames play in the formulation and solving of these problems. They also confirm that the three dimensions of didactical functionalities are already present in the design of tools as had been hypothesized. Thus we did not feel the need to revise at this stage the first version of the ITF."

This part of WP1 work carried out in the second phase of the project did not thus introduce modification to the first version of the ITF. The work on WP1 was nevertheless not limited to the establishment of connections between WP1 and WP2. An important part of it was directed towards the preparation of the experimentation, thus the connection with WP4. We briefly review this work and its outcomes below.

Connection with WP4

As recalled in the introduction, ReMath project was built as a cyclic process, and the ITF had to be tested through a series of cross-experimentations which, at the same time, were asked to contribute to the understanding of meaning-making through representing with digital media. From this point of view, each development or extension of DDA was obeying its specific logic and, in the experimentation of its own DDA for instance, each team had specific questions in mind. There was thus a risk of fragmentation of the different experiments that could make difficult the integration of their results into a coherent testing of the ITF.

The connection between WP1 and WP4 was thus crucial. In the deliverable D9, two methodological tools especially elaborated for this connection are presented: the *Protocol of experimentation* and the *Profile of experimentation*. The *Protocol of experimentation* is meant to collect information on experimental designs, and is organized according to the following list of items, the teams being asked to answer only the four first ones at this preliminary stage:

General issue(s) addressed
Theoretical framework(s) adopted
Research Questions related to the general issue(s) through the TF(s)
Clarification of the relationship between RQs and TFs, and between RQs and general issues
Methodology: population, context, description of the activities
Data which will be collected
Explanation of how the analysis of such data can answer the addressed RQs

Table 3: Protocol of experimentation

The "*Profile of experimentation*" is meant to be a tool for supporting the integration and the comparison of the results of the local experiments. It is a rephrasing of the parts 1 and 2 of the ITF specified for the design of each particular experiment.

The results of a first analysis of these protocols and profiles are presented in the deliverable D9. They tend to show that the notion of concern is not discriminative enough, almost every team showing high priority for most concerns. Considering this phenomenon, the Siena team in charge of WP4 introduced the notion of top concern for labelling concerns whose mean value was greater or equal to 4.5. Six top concerns thus emerged, four concerns explicitly dealing with representations and semiotics as could be expected, and beyond these the epistemological concern in the identification of educational goals, and the concern about the functions to be given to the artefact and their possible evolution in the modalities of use. Despite these similarities, this first analysis also shows evident differences between the teams, and between the answers of the same team for different DDAs. Only one concern (related to the characteristics of the DDA) is rated more than 4 in all the profiles, namely the concern about "the ways representations can be acted on" which is given the highest rate in all the profiles. Another fact questions the notion of concern or at least the way it has been used by the teams in their answers. Research questions articulated in the protocols and concern hierarchies and patterns cannot be easily matched. Moreover, links between research questions, concerns and theoretical frames remain implicit in most experiment protocols.

The questions raised by this first analysis of the protocols and profiles of the experiments were dealt with in the experimental phase of the project and the results reported in the associated deliverable D11 and D13. The deliverable D9, for its part, ended with a vision of what could be a final version of the ITF complementing where the methodological tool built so far by:

- "an agreed terminology together with some basic principles which could serve as a common basis for integrative work,
- networking graphs that would visualise possibilities of networking between approaches and theoretical frames beyond this core basis, possibly organized around concerns,
- examples of deeper but partial integration between frames with illustrative instances produced by using the results of the experimentation."

II.3. The cross-experimentations

As expressed in the ReMath Project Proposal, the aim of cross-experimentations was three fold, and not just limited to the test of the initial version of the ITF:

1. providing the validation of the D.D.A.'s both in respect to their functioning as didactical tools and the consistency of such functioning in relation to the theoretical assumptions, on which scenarios (tools and activities) have been designed" (ReMath Project Proposal part B, p. 59);
2. producing common results that enhance "our understanding of meaning-making through representing with digital media" (ibidem, p. 9);
3. "providing validation for the [integrative theoretical] framework" (ibidem, p. 8).

The difficulties mentioned just above led ReMath partners to introduce new methodological tools for improving the cross-experimentation process and ensuring that the results of its implementation would be presented by ReMath partners in a way facilitating the comparison and connections aimed at. Two important decisions were thus taken. The first one was to agree on some main assumptions that would converge into a shared idea of representation and constitute what was called a *Minimal Theoretical Framework* (MTF) on representation. The second one consisted in the articulation of a *Common Research Question* (CRQ in the following) that each team was asked to accept and rephrase according to its specific sensitivities and theoretical frameworks.

The MTF was made of the following three assumptions presented in D11 as a skeleton of an implicitly shared theoretical framework concerning representation:

1. "No direct access to mathematical objects is possible, rather mathematical meanings are represented through language, formal mathematical notations, informal idiosyncratic representations;
2. Representations play a fundamental role in the "generation" of mathematical meanings, and this role is assumed to be crucial in the teaching/learning of mathematics;
3. Digital artefacts can provide representations of mathematical objects with a clear potential of generating mathematical meanings;"

together with a description of the idea of representation in the following terms:

> "A representation can be seen as a relationship between:
>
> - a representing, i.e. something with a perceivable nature, accessible to one's senses, and
> - a represented, i.e. something which is not accessible to one's senses but which is considered as existing in some sense (e.g. an idea, a concept, a process...).
>
> but, that such relationship does not exist in itself, it is such only from someone's perspective. That is to say that a representation is such only if there is someone who recognizes a relation between something perceivable – a representing – and a corresponding represented."

It was added that this perspective taken on representation entails that individuals can share representations only through sharing the perception of the perceivable representing. And it was also acknowledged that it may happen that in spite of common perception of the representing, individuals fail to share the represented.

Moreover, as mentioned above, a Common Research Question was articulated. This CRQ was the following: *"How can the representations identifiable in the DDAs be put in relationship with the achievement of specific educational goals?"* ReMath teams were asked to reformulate it for each teaching experiment carried out according to the specific theoretical framework assumed to frame their own investigation. The key idea was that on the one hand the different theoretical frameworks should provide means to refine the Common Research Question, articulating the general common issues according to different theoretical tools, but preserving its essence, expressed in the MTF. Beyond that, ReMath teams were also offered the possibility of articulating specific research questions, and were of course expected to also link these to their key concerns and theoretical frameworks.

All along the process, as reported in DEL 13, these methodological tools were further elaborated for supporting ReMath experimentation and making it productive. The initial Profile of Experiment was refined for better structuring the information and thus potentially facilitating the comparison of results at the end of the experiment on the one hand, and supporting teams in their effort of making clear the various elements in focus on the other hand. This refinement gave birth to the *Research Profile of the Teaching Experiment*. Considering the limited discriminating power of the language of concerns as initially used, such Profile was meant to make it possible to isolate "*research concerns*" from other more general concerns that could inspire at large the teaching experiments. Team agreed that, when filling the Research Profile, they should express only their research concerns. Consistently, each team should formulate Research Questions related to its own *top research concerns*; the link among research questions and related concerns should be made clear as well. Moreover, UNISI team leading WP4 added the explicit request of analytically specifying the theoretical construct related to each concern.

Following the Teaching Experiment Guidelines, each team eventually produced a Teaching Experiment Portrait for each Teaching Experiment it developed. In particular, those documents contain (a) a specification of the educational goals of the Pedagogical Plans implemented, as well as of the didactical hypotheses underpinning them, and (b) a formulation of the different Research Questions (both Reformulation of the Common Research Questions and possible Specific ones) addressed through the Teaching Experiments.

Other tools were then elaborated for structuring the presentation of the results of the different Teaching Experiments so as to facilitate the comparison of the results themselves. Once again, the need was felt of both referring to a common language and using a common format for communicating the emerging results. As explained in D13:

> "We decided to adopt the language emerging from the construct of Didactical Functionality, the notion of Representation, the Minimal Theoretical Framework and the Integrative Theoretical Framework. We wanted to communicate our results and convey ideas in a way to facilitate our reciprocal understanding, without renouncing to our own theoretical frames, not losing the richness of difference. Thus we developed a shared format for presenting our results. All that brought to the set up of two distinct methodological tools: the Teaching Experiment Synthesis Frame, and the Teaching Experiment Analysis Guidelines."

The Teaching Experiment Synthesis Frame was intended to give a common structure for summarizing both the main results and the evidence supporting those results. Short summaries of the TEs were also expected to provide that context of information needed to make sense of the other teams' answers to their own Research Questions. The Teaching Experiment Analysis Guidelines provided a common structure to frame the answers to the Research Questions addressed through the TEs and formulated in the TE Portraits.

The results of this complex enterprise are presented in D13. Quantitative and qualitative data about the experimentation are first presented. These show that each DDA has been substantially experimented but also that, in many cases, the implemented versions of the pedagogical plans were not those initially designed. The efforts made for understanding the reasons for these changes highlight the impact of contextual factors. Their precise role is further investigated through specific questionnaires. Regarding the ITF, the results of these questionnaires confirm the interest of distinguishing between the global and local dimensions of contexts. DEL13 focuses then on the results of the teaching experiments regarding the two first aims of the ReMath cross-experimentation: the validation of DDAs and the improvement of our understanding of meaning-making through representing with digital media. The test of the ITF is thus in some sense indirect in the analyses. Never the less, these provide interesting insights that we briefly summarize below. These come especially from the last part of D13 where a first attempt of cross-analysis, which has been then systematized and deepened in the last year of the project, is made.

As made visible above with the description of the methodological tools supporting cross-experimentation and the rationale underlying these, the design and the analysis of cross experimentation have their roots in the two theoretical tools set up in WP1: the construct of Didactical Functionalities, and the system of Concerns. In what follows, we summarize thus the result of the indirect test of these constructs provided by the cross-experimentation.

The interest of the construct of Didactical Functionality (DF) for making more explicit the design choices and the rationale underlying these, for making visible theoretical similarities and differences and questioning the influence of these on both the design and the analysis of the teaching experiments, is confirmed by the cross-experimentation. The way the construct of DF fosters the expression of links between Educational Goals (EG) on the one hand, and both DDA characteristics and modalities of use (here in form of pedagogical plans) on the other hand, results especially helpful for making sense of the different Teaching Experiments (TE) and for comparing them. Investigating the degree of proximity of the EG expressed by different teams for the same DDA for instance seems a productive entrance for better understanding the combined influence of contextual and theoretical characteristics on the perception and expression of DF.

Through the analysis in terms of DF, interesting differences have been made evident with respect to epistemological and pedagogical approaches according to different conceptions of 'learning', of 'teaching', of what means "to have a certain knowledge or concept", and to the use of different modalities for recognizing the achievement of a particular educational goal. D13 provides thus evidence that the constraint of using the categories of DF imposed to ReMath teams resulted a productive one and constituted a strong element of uniformity*:* differences seemed to be *filtered* by the frame of DF and because of this filter they assumed a consistent form and became workable.

As regards the language of concerns, it showed its limits as a meta-language which, in spite of its usefulness, remains too polysemic for supporting efficiently enough the discrimination of theoretical orientations. The case of the semiotic concern is especially illustrative. It was a top common concern, what could be anticipated considering the aim of the ReMath project. All the teaching experiments shared thus a common focus on the representation of mathematical objects offered by the different DDA and all provided a positive answer about the didactic potential offered by the different DDA with respect to their different representations' systems. Never the less the different theoretical approaches used by ReMath teams have different modalities to explain/describe the reference between the DDA – specific elements and specific way of using it - to the mathematical object. Each team uses different theoretical tools to approach the functioning of the representations available in a DDA (and the related semiotic processes) in relation to the achievement of the specific educational goal. Not all theoretical frameworks used by the teams consider semiotic processes in the same way. In particular, not all theoretical frameworks have a model of the semiotic process available, and when it is the case, not all these models present the same elements and the same level of detail. In fact, in spite of the fact that all the teams declare a high semiotic concern, not all of them have specific theoretical tools for semiotic processes. There is thus no doubt that the meta-language of concerns needs to be seriously reworked if we want it to become an effective tool for supporting theoretical comparison and integration.

Beyond that, the way the different teams provided their answers to the Common Research Question, the criteria they expressed for evaluating the success of their cross-experimentation, showed evident contrasts. In some cases precise outcomes or effects were expected and the validation was made considering the evidence provided by the experimentation in that respect, in other cases expectations were expressed in much more open terms and positive evaluation resulted more from the evidence of the existence of meaning-making processes through the interaction with the DDA. Though almost everybody talked about meanings, some of the research teams gave only a generic reference to "construction of meanings", while other approaches provided detailed descriptions of "evolution of meanings through specific semiotic processes". Comparing and coordinating such different visions of results was difficult, as pointed out in D13. Even if a better understanding of the differences had emerged at that stage of the analysis of the cross-experimentation, further advances needed to be made for their productive exploitation in the ITF.

These results and analyses led ReMath teams to engage in a systematic comparison of the familiar and alien experimentations attached to each DDA, what was called the cross-case analyses. The results of these cross-case analyses and their implication for the ITF will be presented further in this deliverable.

III. Situating ReMath work with respect to the international reflection

In the previous WP1 deliverables, we have pointed out that ReMath concerns regarding theoretical fragmentation were increasingly shared within the international community of mathematics education. For that reason, we find important to situate ReMath theoretical reflection and work with respect to the progression of this international reflection. This increased interest is especially visible in the last conferences of the European Association for Research in Mathematics Education (CERME), especially in the contributions to the working group on theoretical comparisons and in the discussions generated by these contributions[2]. These have resulted in a special issue of the Zentralblatt für Didaktik der Mathematik (Prediger, Arzarello, Bosch, Lenfant, 2008) which complements the two special issues published in 2006 on theoretical issues in mathematics education, which were considered in the deliverable D1. In our opinion, this special issue represents in a fairly accurate way the current state of the international reflection, and of the type of attempts developed in order to overcome theoretical fragmentation. This is the reason why we have chosen to especially situate ReMath advances and work with respect to it in this deliverable. We nevertheless complement the vision given by this special issue by that emerging from the chapters focusing on theory of the recently published Second Handbook of Research on Mathematics Teaching and Learning (Lester, 2007), a preprint of one of these chapters by Cobb having also been already evoked in the deliverable D1.

We first present the vision of theories underlying this special issue and the handbook and introduce the vision of networking between theories developed in the special issue. We then summarize the strategies for theoretical networking that emerge from the contributions to the

[2] Members of ReMath have regularly participated to this working group and contributed to the special issue.

special issue and the methodologies associated with these strategies. We finally synthesize the main lessons we draw from this special issue, situating these with respect to the ReMath perspectives and advances.

III.1. A vision of theories and of networking between theories

In the introductory article to the special issue, entitled "Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework", Prediger, Bikner-Ahsbahs and Arzarello point out first the heterogeneity of what is called a theoretical framework or a theory by different researchers and different scholarly traditions, from basic research paradigms like socio-constructivism or interactionism, comprehensive general theories like the theory of didactic situations or the anthropological theory of didactics, to local conceptual tools such as the modeling cycle. In order to make sense of this diversity, they introduce a distinction between a static and a dynamic view of characterizing the notion of theory:

> "For analytical reasons, we distinguish two ways of characterizing the notion of theory in the following considerations.
>
> - A normative more static view which regards theory as a human construction to present, organize and systematize a set of results about a piece of the real word, which then becomes a tool to be used. In this sense a theory is given to make sense of something in some kind and some way (for instance Bernstein's structuralist perspective, discussed by Gellert, 2008).
>
> - A more dynamic view, which regards a theory as a tool in use rooted in some kind of philosophical background, which has to be developed in a suitable way in order to answer a specific question about an object. In this sense the notion of theory is embedded in the practical work of researchers. It is not ready for use, the theory has to be developed in order to answer a given question (for example, most researchers who follow an interpretative approach adhere this dynamic view on theories.)"

The diversity of interpretation given to the term of theory in mathematics education has been already stressed by many researchers. In the chapter entitled "Theory in mathematics education scholarship" of the handbook mentioned above, Silver and Herbst approach this issue emphasizing the historical influence on this phenomenon of the visions of theory carried out by the diverse fields that have been progressively connected to mathematics education and have influenced its development, from psychology and mathematics to linguistics, anthropology and semiotics. Adopting an instrumental view on theories, they then structure their discussion about theories by examining how these might serve as a mediator of the dyadic interactions between the three vertices of what they call "the *scholarship triangle*": Research, Problems and Practice, what leads to a list of categories illustrated by many examples. In the final part of their chapter, coming to the role that theory could play for strengthening the field as an endeavor of academic scholarship, they introduce two new categories, going beyond the *local theories* that help mediate specific dyadic connections that they have considered so far. These are on the one hand the category of *grand theories,* and on the other hand the category of *middle-range theories.* According to them, grand theories "respond to a need for broad schemes of thought that can help us organize the field and relate our field to other fields" (p.60), and also aggregate scholarly production within the field. The

Anthropological Theory of Didactics (ATD) is for them an example of a theory having such an aim. Middle-range theories in reference to the sociologist Merton respond to "the need to inform a discrete variety of practices, including individual mathematical thinking, teaching and learning in classrooms, or mathematics teacher education" (p.61). These are presented comparable in scope with mathematics theories such as group theory, "which identify subfields of study and the specific methodologies used to study those" (p.61). According to the authors, the field of mathematics education should pay "considerable attention to middle-range theories that conceptualize patches of reality and guide empirical research, bridging the gap between grand theorizing and empiricism." (p. 61). They cite the theory of instructional situations developed by Herbst and Brach (2006) augmented by Brousseau's notion of didactic contract, as an example of such theory.

In his conclusive chapter to the same handbook, Niss comes back to this issue of polysemy of the concept of theory, and while acknowledging the interest of the position defended by Herbst and Silver, risks another definition:

> "A theory is a system of concepts and claims with certain properties, namely
>
> - A theory consists of an organized network of concepts (including ideas, notions, distinctions, terms, etc.) and claims about some extensive domain, or a class of domains, consisting of objects, processes, situations and phenomena.
> - In a theory, the concepts are linked in a connected hierarchy (oftentimes, but not necessarily, of a logical or proto-logical nature), in which a certain set of concepts, taken to be basic, are used as building blocks in the formation of other concepts.
> - In a theory, the claims are either basic hypotheses, assumptions, or axioms, taken as fundamental (i.e., not subject to discussion within the boundaries of the theory itself), or statements obtained from the fundamental claims by means of formal (including deductive) or material (i.e., experiential or experimental with regard to the domain(s) of the theory) derivation." (p. 1308)

According to Niss, theories may serve six different purposes: explanation of some observed phenomena, prediction, guidance for action or behaviour, providing a structured set of lenses, a safeguard against unscientific approaches, protection against attacks from outside. He points out that the majority of the theory actually invoked in mathematics education research are borrowed from other fields and "are of too general a nature to be transposed to offer a sufficient pool of specific results and concrete methodologies, so as to provide complete guidelines for research in our field" (p. 1309). He acknowledges the existence of some examples of "home-grown theories" but, according to him, these are of too limited scopes to provide a comprehensive coverage of the field. He then articulated what a theory of mathematics education should contain. It should contain at last the following categories that should be integrated into a coherent and consistent whole:

- "A sub-theory of mathematics as a discipline
- A sub-theory of individuals' and groups' affective notions
- A sub-theory of individuals' and groups' cognitive notions
- A sub-theory of the teaching of mathematics

- A sub-theory of teachers of mathematics" (p. 1310)

He adds that for each of these five domains, several meaningful sub-theories could probably be created, which combined would give rise to competing grand theories for the field. While acknowledging that the search for a grand theory of mathematics education can be seen as futile in the current state of development of the field, he plaids for what he calls a meta-research program of theory archeology, explained as below:

> "By this I mean a systematic effort to identify, uncover, and analyse the alleged as well as the actual roles of "theory", "theoretical framework", and "theorizing" encountered in a wide selection of publications in mathematics education research. " (p. 1310)

Prediger, Bikhner-Ahsbahs and Arzarello explicitly situate themselves with respect to the vision expressed by Niss of a theory as an organized structure of concepts, seeing it too much restrictive and static for being appropriate to their purpose. Their position is to accept diversity of theories as a richness, and not to exclude some of the traditions they want to connect.

This leads to the following vision of what a theory is:

> "In our understanding, theories or theoretical approaches are constructions in a state of flux. There are more or less consistent systems of concepts and relationships, based on assumptions and norms. They consist of a core, of empirical components, and its application area. The core includes basic foundations, assumptions and norms, which are taken for granted. The empirical component comprise additional concepts and relationships with paradigmatic examples; it determines the empirical content and usefulness through applicability." (p.169)

While accepting diversity as a richness, Prediger & al. acknowledge that this diversity can only become fruitful when different approaches and traditions come into interaction, thus when connections are developed. For labeling such connections, the authors use the word "networking" (also introduced in ReMath deliverable D1). With this term, they want to express connecting strategies "that respect on the one hand the pluralism and/or modularity of autonomous theoretical approaches but are on the other hand concerned with reducing the unconnected multiplicity of theoretical approaches in the scientific discipline" (p. 170).

III.2. Strategies and methodologies for theoretical networking

Strategies

The special issue proposes indeed a categorization of the different ways of dealing with the diversity of theories. These ways are positioned along a linear scale according to their degree of integration. At the two poles of the scale, one finds the two extreme positions: ignoring other theories, unifying globally, while the intermediate positions label different strategies for networking theories:

- understanding others and making our own theories understandable,
- comparing and contrasting,
- coordinating and combining,
- synthesizing and integrating.

The special issue only considers the four intermediate positions, the extreme one, unifying globally, being discarded for two fundamental reasons. The first reason is that searching for global unification does not seem compatible with the vision shared within the CERME working group that a diversity of theoretical lenses is necessary for grasping the complexity of the educational reality. The second reason is that there exist incompatibilities between existing and influential theoretical approaches in the field, for example regarding their basic assumptions on learning. In such conditions, it seems difficult to achieve a coherent unification without abandoning core assumptions in some of these theories. This being said, let us consider the intermediate positions.

Understanding and making own theories understandable is presented as a first and necessary step towards a networking of theories, inherent to any attempt of communication beyond a local community. Hence its position along the integration axis. Comparing and contrasting is the second couple which also serves the cause of better mutual understanding. In the contributions published in the special issue, this corresponds to the mostly used strategy, but the criteria used offer evident variation. Prediger & al. classify these into three main categories: the role of well chosen implicit or explicit aspects in the theoretical structures such as conceptualization and role of social interactions, the articulation of the respective theories in the practices of experimental level (in the conceptualization of research problems or phenomena, in the analysis of a given piece of data for instance), and a priori defined criteria for quality of theories (validity versus relevance, degree of explicitness, empirical scope, connectivity…). With coordinating and combining, a new phase is reached. Comparisons and contrasts are always possible but coordinating and combining supposes some level of compatibility between the theories at stake. Half of the ten papers published in the special issue reach this stage, using complementary lenses for analyzing some concrete empirical phenomenon or even building some conceptual framework by fitting elements from different theories. The TELMA contribution (Cerulli & al., 2008) corresponds to the first category while the contribution by Arzarello & al. combining the idea of semiotic bundle and semiotic game coming from APC and the idea of praxeology and the ostensive/no-ostensive dialectics coming from ATD for explaining the phenomenon of chirographic reduction seems closer to the second one. This networking strategy in terms of coordination and combination, if systematized, can be expected to lead to the development of new pieces of synthesized or integrated theory. Coherently with their position, Prediger & al. only envisage such networking strategy at a local level. As they explain, "the notion synthesizing is used when two (or more) equally stable theories are taken and connected in such a way that a new theory evolves. But often, the theories' scope and degree of development is not symmetric, and there are only some concepts or aspects of one theory integrated into an already more elaborate dominant theory" (p.173). In the special issue, only one contribution refers to such networking strategy, that of Steinbring analyzing the development of his epistemological perspective on social interactions (Steinbring, 2008).

Considering again the Second Handbook of Research on Mathematics Teaching and Learning, it is interesting to connect these categories with those introduced by Cobb in the first chapter entitled: "Putting philosophy to work. Copying with multiple theoretical perspectives" already mentioned in the deliverable D1. In this chapter, having in mind a vision of mathematics education as a design science, Cobb situates within a comparative perspective and introduces two main criteria for comparing theoretical approaches: how they conceptualize the individual, their potential to contribute to our understanding of learning processes and the means of supporting their realization. These two criteria can be respectively related to the first and third categories of criteria for comparison presented above. Cobb then uses these criteria for contrasting four theoretical perspectives: Experimental psychology, cognitive psychology, sociocultural theory, and distributed cognition, emphasizing that "each of these perspectives is a tool that has been fashioned while addressing problems that are not of immediate concern to most mathematics educators" (p. 29) and have thus to be adapted to the concerns and interests of mathematics educators. These four theories are very general theories and definitively not home-grown theories for the field of mathematics education. Referring to Lawler (1985), Cobb plaids for an adaptation consisting of some kind of *bricolage* : "A *bricoleur* is a handy man who invents pragmatic solutions in practical situations, adapting ideas from a range of theoretical sources". He then illustrates this idea of *bricolage* by the framework he has been developing with colleagues over a number of years, seen as a bricolage between social and cognitive approaches, the social perspective drawing on sociocultural theory, the cognitive perspective on both cognitive psychology and distributed accounts of cognition. In the networking language we use here, there is no doubt that such a construction belongs to the fourth category, that of synthesizing and integrating, and the balance reached in it between several mature theories leads to consider that the term synthesizing could be more appropriate than the term integration. More generally, Cobb sees such *bricolage* as the way we can cope with competing and incommensurable theoretical perspectives as is the case in mathematics education. And coherently with Prediger & al., he considers that "the primary challenge posed by incommensurability is to develop a way of comparing and understanding different perspectives" (p.31). The criteria he has selected and used try to answer this need, but as he fully acknowledges, these are only one system among many possible others. He emphasizes that there does not exist a neutral framework for comparing incommensurable theoretical perspectives but that this absence does not condemn us to absolute relativism.

Methodologies

Beyond the identification of networking strategies, it is important to consider the methodologies used in the different contributions of the special issue for carrying out the networking between theories. These are quite diverse. For instance, for comparing theories:

- Bergsten investigates how theories influence research on teaching and learning limits of function by using the scholarship triangle of Silver and Herbst already mentioned for comparing different research studies in that area.
- Bosch & al. reformulate the problem of metacognition into ATD, showing how this conversion from a cognitive perspective to the ATD perspective changes its characteristics, and using this analysis for contrasting more globally cognitive and institutional perspectives.
- Steinbring analyses the historical development of his epistemological approach of social interactions, looking for moments of bifurcation in this development, investigating the underlying reasons and using this for comparing and contrasting the different theories involved.

- Prediger compares theories by analyzing the ways researchers coming with different theoretical backgrounds transform a series of questions coming from practice into research questions.
- Kidron & al. compare three theories through the way these theories approach social interaction, using comparison by pairs of theories and investigating what each theory might learn from the other.
- Cerulli & al. build a complex cross-experimentation process from which the ReMath methodology emerged for understanding and comparing the role played by different theoretical frames in design.

It is interesting to notice that when combination and coordination is at stake, what is combined are generally theories with different grain-sizes which are put in relation of complementarity. For instance, Arzarello & al. try a local coordination of APC and ATD, considering their different focus (cognitive/institutional), time and grain sizes (micro/macro). This is also the case for Gellert who analyses the same data from two different perspectives: Bernstein's structuralist perspective and the interpretative approach, and tries to combine these analyses, relying on the different roles and grain sizes of the respective theories.

From a methodological point of view, it seems also interesting to contrast two very different cases. In the first case (7 contributions), the contribution is proposed by a single researcher or several researchers sharing the same background. In the second case which corresponds to the ReMath situation, the contribution emerges from the collective work of researchers with different backgrounds. There are only four contributions having this profile, that of Arzarello & al., Cerulli & al., Kidron & al., Prediger. All of these only emerge from two groups: TELMA on the one hand, and an informal group created at CERME 4 on the other hand, which shows that such collaborative work of theoretical networking mixing educational cultures remains quite limited. It is also interesting to notice that, in order to develop its networking activity, the informal group mentioned also felt the necessity of going beyond exchanges and discussions of selected publications, and of developing a specific methodology. Transcripts from one Italian classroom session complemented by a questionnaire to the teacher, answers from the different researchers to common questions coming from practice were thus used and progressively designed for making sense of the way the different theoretical backgrounds were impacting research practices, from the formulation of questions to the interpretation of data.

III.3 Outcomes and relationships with the ReMath perspectives

Regarding the precise outcomes of these networking efforts, what is evidenced by the sources used shows that these outcomes are still limited. Of course, the existing theoretical perspectives in mathematics education often result from partial synthesis or integration realized by individuals or communities, as has been evidenced above, but the external networking of well established theoretical perspectives that have developed in different communities, are shaped by the historical development of these communities, have their specific dynamics, is another and much more challenging problem. In the special issue, from this point of view, the contribution which seems the closest to the ReMath project is that by

Arzarello & al. trying to network APC, a cognitive approach based on activity theory and ATD.

What has been recalled from ReMath perspectives in the previous part shows evident resonance with the analysis presented here, and we would like just to stress some important points. We see particular resonance with the attention that the different authors mentioned pay to theoretical issues, pointing out the fragmentation of the field and its negative consequences, and at the same time considering that at least in the current state of development of the field, the search for a grand and comprehensive unified theory is not really pertinent. The networking metaphor with the different strategies it can support, from comparison towards local integration, playing on complementarities while being especially sensitive to the specific coherence of each particular theoretical perspective seems perfectly in line with the positions that we have developed in ReMath. We see also resonance with the attention that seems increasingly paid to the role that theory exactly plays in research practices, the increasing attention paid to their instrumental value, and the vision of these as dynamic entities.

There is no doubt that the theoretical perspectives on which the ReMath teams rely offers a great diversity in terms of basic assumptions, focus and grain-size, that they have also different degrees of maturity. There is no doubt also that these theories even if partly stabilized are still evolving, and that the technological evolution is one of the forces piloting their dynamics. Most of these theories already result from some integrative (or synthesis) work carried out often in the long term inside a given community. The challenge for ReMath is to go a step further, finding the ways for networking theoretical perspectives which have developed more or less independently in different European communities, and in this networking process going as far as possible while being faithful to each of the theoretical perspectives at stake. For that purpose, ReMath building on the TELMA experience has elaborated a sophisticated methodology of cross-experimentations complemented then by cross-case analyses. Looking at the existing literature at the end of the ReMath project we can see that, at the European level, this is still at the moment a unique experience if we consider the number of teams and educational perspectives involved, the number and length of the experimentations carried out, the intertwining of design and DDA development, and the sophistication of the global process.

In the following, as announced, we will try to draw lessons from this project and delineate what are its exact outcomes in terms of theoretical integration.

IV. Building on DDA development: the DDA epistemological profiles

An important specificity of ReMath, considering theoretical integration is the fact that the project has combined the development of DDAs and that of scenarios of use based on a vision of their respective didactical functionalities for the cross-experimentation. Six DDAs were involved. Their diversity certainly limited the ambitions of the project in terms of artefact connection but it was a real chance for reflecting on issues of theoretical integration. The diversity of the ReMath DDAs is multi-dimensional, and only partially linked to differences in theoretical perspectives. Mathematical domains at stake vary from one DDA to another.

Moreover some DDAs only refer to mathematical concepts while others link these with other fields such as geography or physics. All DDAs offer some novelty in terms of representations but the distance of these creations to the representations usually dealt with in educational systems varies a lot. The possibilities of evolution of the representational systems they implement also vary substantially from one DDA to another. At the same time, similarities exist. Two DDAs focus on elementary algebra, two situate more at the interface between algebra and analysis while two finally focus on 3D space and geometry. All DDAs connect different systems of representation in an original way, and many of them link some kind of algebraic symbolism and geometry. They form thus a particular rich landscape for exploring the needs and possibilities of theoretical integration in that area. In order to better make sense of this diversity and the way it impacted our theoretical reflection, and also for facilitating the communication of ReMath outcomes towards a wider audience, in the last stage of ReMath, at the time the development or extension of DDAs was completed, it was decided to create for each of the DDAs an epistemological profile. We present below the common structure adopted for these epistemological profiles, then synthesize their affordances. The details of the epistemological profiles are given in Appendix 1.

IV.1. Methodology

The common structure elaborated for the epistemological profiles is organized into four different dimensions: objects, connections, activities, pedagogical/intervention agenda. For each dimension, the nature of the information to be provided is detailed.

Objects

What are the main objects represented and manipulated by/with the DDA?

Representations:

For each object, describe how they are represented and manipulated by the user. Please distinguish if relevant:

- Dynamic / Static.
- Mathematical / Computational.
- Direct Manipulation / Menu driven.
- Compatible with the curriculum / Innovative

Can the user (teacher or student) build representations of objects ("half-baked" objects)?

Concepts :

For each object, please indicate the main associated mathematical concepts and the way the representation deals with.

Indicate whether the concepts are rather strictly mathematical or whether the DDA opens to broader concepts (e.g. in physics, geography…)

Connections

Please describe how the objects are connected.

If possible describe these connections in terms of semiotic systems or semiotic chains.

A map with commented arrows could also be useful.

Activities

What kind of activities does the DDA aim at:

- Proof
- Exploration
- Modelling
- …

Explain how the objects and connections favour this kind of activities.

Are these activities compatible with existing curricula and practice, or are they innovative?

Pedagogical/intervention agenda

Situate the DDA design with respect to the following dimensions in terms of pedagogical versus intervention agenda:
1. Innovation / Acceptance
2. Distance to traditional curriculum
3. Status of representations: traditional, innovative, to be handled, related to others, etc.
4. Scope for mathematization or only about mathematics

IV.2. Synthesis of the epistemological profiles

This synthesis is structured along six different themes which, according to us, help to make sense of the diversity of ReMath DDAs, of their potential for mathematical learning, in connection with the theoretical visions of representations and learning processes underlying them. After a brief description of ReMath DDAs, for each theme, a summary of the characteristics of each DDA is presented and followed by a horizontal view of the six DDAs. The six themes are:

- What mathematics are foregrounded and emphasized most to the user?
- Diversity and Connections between representations
- Ways representations are manipulated
- Relationship to standard systems of representation “innovation/acceptance”
- Theoretical visions of representations, and of digital artifacts.
- If not developed with Web “2.0” in mind, how might it be accommodated?

Summary of DDA

Aplusix

This environment is an open-ended microworld for: (1) converting between “usual” algebraic expressions and equations, “tree” representations which represent the expressions and equations with nodes of operands and operators, and “hybrid” representations that can contain subalgebraic components, in addition to single operators and operands, as nodes; (2) doing calculations using “tree” or “hybrid” representations.

Casyopee

The environment is oriented towards problem-solving, modeling and proof with mathematical functions, and offers special assistance for this activity. It includes two connected windows: a symbolic and a dynamic geometry (DG) window. The former allows users to create functions, defined on |R or some user-defined subset of |R, plot those functions, perform manipulations, calculations and proofs with those functions, and include them as components of the (DG) window. In the latter, the user can explore dependencies in geometrical figures, express them as functions between magnitudes and export towards the symbolic window.

Alnuset

Alnuset includes three connected representational environments: a one-dimensional number line, an algebraic manipulation toolkit, and a two-dimensional Cartesian plane. Users can explore algebraic expressions and functions and their component parts in the context of these one- and two-dimensional number lines by defining parameters that can be dragged along the line and expressions involving those parameters that adjust automatically. On the one-dimensional number line, these "constructions" (i.e. expressions consisting of user-manipulable parameters along the number line) also exist on the number line and their values are determined using the Thales and parallelogram theorems, for which illustrations on the number line can be turned on by the user. Finally, users can define propositions that can be explored by dragging parameters along the number line to determine when those propositions are true or false, and number sets can be constructed to define truth-values for given propositions.

Users can import expressions, equations, and inequalities defined in the number-line environment into the algebraic manipulator (or construct new algebraic objects within the manipulator) and then explore ways to re-represent those expressions by clicking on different components of an algebraic equation or expression (for example, in the expression ((x+y)/4) users can click on the entire fraction, or the (x+y) component, the x or y alone) and the manipulator indicates the sorts of mathematical transformations that can be done without changing the value of the expression (for example, (x+y) can be multiplied by 1, added to zero, divided by 1, and so on).

Finally, the two-dimensional number line (or Cartesian Plane) environment plots the equations provided by users in two dimensions, and operates much like a limited (that is, deals mainly with functions of the form f(x)) dynamic geometry environment where parameters, like in the one-dimensional case, can be dragged and the corresponding expressions and propositions that are built from those parameters change accordingly.

MoPiX

This environment enables the user to animate and otherwise manipulate visual objects by defining functions on them using equations that direct different properties of the object (height, width, Cartesian location, color, size, and so forth) to change in systematic ways. The properties of other objects in the environment can be included as a parameter along with time of animation, so that objects can interact with one another and depend on one another for their behavior. An equation library exists from which preexisting equations that define 'default' or 'common' behaviors can be selected and changed. The animation is executed using standard "play"-style buttons that advance the time parameter while each equation executes and the corresponding properties of included objects update.

MaLT

MaLT is a LOGO-based programming environment built on a 3D games engine for the construction and exploration of 3D geometric objects. Objects are created by either adding existing solids (such as spheres or cylinders) to the environment, or by programming a turtle within the environment to follow programmed LOGO instructions and "draw" the object. The strength and novelty of this environment lies in the user's ability to then dynamically manipulate these objects by varying one, two, or three of the programmed objects' parameters with respect to one another via number line or Cartesian representations. For example, if the user instructs the turtle to draw a rectangle by writing a generic procedure and then instantiating the rectangle with specific parameters:

```
to rect :a :b :c
rr(:c)
 repeat 2 [ fd(:a) rt(90) fd(:b) rt(90) ]
end
rect(10 10 0)
```

They can then access a "2D Variation Tool" where they can dynamically manipulate the co-variation of parameters a and b on the (x,y) plane and see how those parametric updates affect the shape of the rectangle. The user might, for example, trace the shape of the function f(x)=x and notice the rectangle growing both taller and wider, or trace the shape of the function f(x)=-x (plus some constant) and notice the rectangle growing shorter but wider, or wider but shorter, depending on which parameter they had selected to be represented by which axis. The user can also change parameter c either on its own or in connynction with one or both of the others and observe the rectangle rotate along axis z. Another strength of this software is that the user can dynamically manipulate the 'camera', i.e. the point of view of the user to any figure and for example rotate around the figure, move close or even 'inside' it. Finally the user can make choices amongst a set of backgrounds of a variety of visual cues for a 3D effect.

Cruislet

Cruislet is a navigation microworld in which users direct 'avatars' (air transportation vehicles) across the Greek geography by programming stepwise navigation instructions in either graphical/Cartesian or spherical/polar coordinate systems. Avatars' movements are defined in terms of displacement vectors, and must take into account not only location, but also the elevation of the landscape they are navigating. Avatars can be programmed independently, together, or with some functional relationship defined that helps each avatar to interpret similar input commands. Cruislet uses the same Logo language as MaLT but is built on a platform containing geo-coded information and navigation functionalities.

What mathematics are foregrounded and emphasized most to the user?

Aplusix

The mathematics emphasized here is the structure of algebraic expressions. The trees foreground the structural properties of mathematical expressions, helping readers to parse them, to unpack their subcomponents and to reconstruct them. This also emphasizes order of operations and ways of representing operations, as well as “groups” or “terms” in equations and expressions. It might also foreground into the ‘priority’ or role of different terms, it might be that for example a variable or an operation in a “high” branch affects the overall equation more than one in a low branch.

Casyopée

This DDA highlights the importance of mathematical functions as models of dependencies and tools for problem solving. Parameters help to express generality. The construction of representations does not privilege one between the many aspects of functions: geometrical dependencies, curves, tables, graphs, symbolic formulas and domain. Casyopée offers special means to work with the symbolic representation, often an obstacle to beginners, and to make it useful, especially in the proving process.

Alnuset

This DDA emphasizes the different ways that components of an algebraic expression, equation, or inequality contribute to the final value it is meant to represent: specifically, the way that parameters, operations, and the way the operations are grouped construct both a generic representation of a solution, as well as specific instantiations of solutions depending on the values of the parameters. It also provides a nice tool for exploring the variety of different algebraic (‘usual’) representations that can produce the same (specific) output via different sequences of parameters and operations, that can produce equivalent (generic) output via different sequences of parameters and operations, and that are simply identity manipulations (such as adding zero or multiplying by 1). The representations of the Thales and parallelogram theorems is also a novel and flexible way to conceive of the basic operations such as addition, subtraction, multiplication and division (especially the case of multiplication and division with fractional operands).

MoPiX

This environment emphasizes the role of inputs and outputs in mathematical modeling, and the powerful role of functions as representing dynamic events that change over time. Compared to the other DDAs, it is quite parsimonious in that the same two representations (“object” and “function”) can be used to produce a wide variety of digital artifacts that may not be traditionally considered mathematical. When programming objects using mathematical functions, the role of specific inputs and expressions is visually emphasized in the construction kit by highlighting the different components of each equation.

MaLT

The main relationships emphasized by the environment include programmatic mathematization of 3D geometric objects, and the co-variational role of parameters in those programs. Although programming plays an important part in the environment, the foregrounded mathematical relationships are accessed through dynamic manipulation of physical objects in the environment using graphical representations. Geometrical constructions can be viewed as they exist in space by dynamic motion of the user's point of view and location vis-à-vis the object itself.

Cruislet

Cruislet emphasizes the role of vector representations and on the co-existence of Cartesian/geographical and polar coordinate systems. Displacement of mathematical entities with position and direction (in the form of avatars) can be programmed. This links programmatic mathematics to these geometrical systems forging links between algebraic entities such as functional relationships and parameters to 3D space.

Synthetically, the six DDAs span a wide area of mathematics, from algebra to analysis and from Euclidean style Stereometry to coordinate geometry. Although this section is about the mathematics represented in the DDAs and not the representations themselves or the ways in which they are meant to be used and placed in meaning, the DDAs can be placed along the pure – applied mathematics axis. The first three DDAs are closer to the idea of accessing the mathematics as objects of thought. The latter three have the mathematics placed in a specific role, as tools for engineering or effecting changes to mathematical entities. However, this broad distinction does not mean that the DDAs' design presupposes specific educational goals. A DDA that seems to favor more directly a mathematical notion can be also used as a tool for application problem solving and a DDA where mathematics are embedded in a domain of application can be used for conceptualizing a specific notion.

MoPiX and Cruislet also perceive mathematics as a means to understand an area of application, in this case Newtonian Physics and Geography respectively. In that capacity, the mathematics is meshed with concepts and information with respect to the area of application. The mathematics is the tool to create something which is meaningful in that respective area, i.e. animations in the former and displacements – trips in the latter. In that respect, the 6 DDAs contain a range of mathematical objects but are also diverse with respect to epistemology and applicability of the mathematics. In the former three, the question of whether mathematics is a theoretical object of study in itself is more open for shaping in the designed or practiced Didactical functionality. In MoPiX and Cruislet this question is more settled towards mathematics as a tool to understand and control areas of application. In the case of MaLT, it is relatively open in the sense that definitions and constructs are mathematical but shaped towards applicability in that geometrical constructs are manifested as objects rather than representations of ideal objects (e.g. the turtle trace looks like a piece of string which gets very thick when the camera gets close to it).

Diversity and Connections between representations

Aplusix

There are three basic representations "styles": 1) the 'usual' algebraic expressions and equations, 2) 'tree-based' expressions and equations with only single operands and operators for each node, and 3) ‘hybrid’ expressions and equations, where a node can contain a sub-expression.

The Aplusix documents also make a distinction between “free” and “controlled” tree representation modes: free mode allows any string of characters into a node (not just numbers, variables, and operators), and controlled mode only allows: (1) correct operators in internal nodes and single-letter variables or numbers in each leave nodes; (2) correct arities (number of arguments) of operators.

Casyopée

In Casyopée representations are complementary and provide different means for exploring situations and solving problems. Users can define a ‘usual’ algebraic function, explicitly defining the set of real numbers upon which the function operates, possibly with the help of the DDA. They can also directly export a function as a model of dependency between magnitudes in a figure created in the DG window. In this DG window, varied representations of dependencies are available for exploration. The choice of magnitudes as an independent and dependent variables makes the connection between these representations. Graphs of the exported “geometrical functions” are dynamically linked to the movement of a free point in the geometrical figure.

Alnuset

All representations are dynamically hotlinked, so that expressions, parameters, propositions, etc. created or updated in one environment are automatically added to and change within other environments within the system. Manipulations can be performed in any of the representational systems – symbolically, visually, or indirectly through the manipulation of expressions’ component parts.

MoPiX

There are two main representations in this system: visual objects (and their encapsulated properties), and equations/functions defined over time that define how the properties of these objects change with respect to time or other properties of objects in the system. Preexisting equations (such as equations that assign gravitational acceleration or oscillation behavior) can be assigned to objects and manipulated, or new user-defined equations can be created and then assigned to objects.

MaLT

Connections between representations are made via “hotlinked” dynamic visual updates in the case of the “variation tools” that manipulate constructed objects. The connections are between the formal representation of such objects by means of Logo programs and the visual representation of these in space.

Cruislet

All representations including vector-based, programmatic, avatar-and-map based, polar and graphical are ‘hotlinked’ such that changes in one representational system are reflected in another, and such that manipulations can be performed in any of these systems.

Synthetically, there is a great diversity in the representations available in the six DDAs. This point is quite striking in the case of the former three DDAs where the areas of mathematics are rather similar addressing algebra and analysis while the representations afforded in each one have significant differences. In Aplusix for example, the connection of the standard representation of expressions and equations to the tree representation in a coexisting way invites users to generate meanings through reciprocal use of the two representations. Each representation affords a different view of algebraic expressions respectively a procedural and a structural view. This connection is therefore part of the didactical design of the DDA.

In Casyopée, the representations are complementary and the emphasis is on the ways in which expressions are created and manipulated. The connections to dynamic geometry provide another means to create expressions and dynamically observe constants and variants. In Alnuset, we have a combination of these two approaches. The expressions are manipulated but also coexist in three very diverse representation systems, one of which (the number line) is innovative providing new opportunity of mathematical meaning making through the forging of connections between the number line and the other representations.

In MaLT, representation of mathematical expressions is manifested through a programming language and is connected to graphical representations which appear strictly as results of executions of programs. These two are connected by means of variation tools which enable dynamic manipulation of parameter values and the resulting dynamic change of the graphical representations. In MoPiX idiosyncratic equation expressions (characterized by time) are connected to the behavior of graphical representations of physical objects and the connections between them. In Cruislet programmatic mathematics is used to navigate avatars in geographical space. This representation co-exists with stepwise displacements made possible by means of representations of geographical/cartesian and polar values.

It is important to consider that the choice of representation shapes the features/characteristics of the tool, a first level in the didactical functionalities of each respective DDA and orients the modalities of uses. For instance, programming is necessary in Malt or Cruislet and direct manipulation is favored by Alnuset. The great variety of choices with regard to representations is thus to be seen as a potential richness for addressing a big range of educational goals.

Ways representations are manipulated

Aplusix

The main manipulations concern nodes and branches: Nodes can be filled, branches can be added. In addition users can add comments. A "Keyboard" shows all possible operand/operation types. Aplusix includes a feature to calculate written expressions into single operands in nodes (for example, highlight "4+3" and the calculate feature will replace with "7") as well as features to automatically convert between usual and tree representations. The translation/connection between representations mainly takes the form of an activity completed by the student, and verified via pre-established activity sequences.

Casyopée

Once functions have been input into the environment in "usual" algebraic form, they can be analyzed in a number of ways by the system to produce new functions (i.e. they can be factored, differentiated or integrated, and so on); they can be graphed; if they include variable parameters those parameters can be manipulated and the corresponding change in the graph examined, the functions' domain can be changed, multiple functions can be compared, solved in terms of one another, and so forth.

The units of manipulation and construction in this environment are matched to mathematical tasks: whenever a calculation or manipulation occurs in the system, the result of that manipulation or construction is saved as a new manipulable object in the system (for example, a new interval on the real number line, a new function, a new specific instantiation of a previously generic function, or a new parameter). Furthermore, these primarily algebraic and functional objects are connected to graphical and geometric construction elements through the two plotting options: Cartesian graphs and a dynamic geometry environment where additional line segments, circles, and other geometric objects of interest can be added.

Alnuset

Via drag and drop (in visual line-based representations), an equation editor, the algebraic manipulator (which makes available for different components of an algebraic expression, equation, or proposition all manipulations that produce equivalent objects), or by directly typing the expression.

MoPiX

Objects' location can be manipulated via drag-and-drop within the object panel. Object properties are manipulated by dropping preexisting equations that define properties of the object (that might use time or the properties of other objects as parameters) onto the object from a library located underneath the object panel, or by writing new equations to control properties of the object in the equation editor located above the object panel and dropping them onto the object.

MaLT

Physical objects are defined programmatically, instantiated by calling these programmed procedures, and then manipulated and investigated via direct manipulation of the parameters involved in those programs through a graphical representation (number line, Cartesian graph, or set of Cartesian graphs).

Cruislet

Avatars can be programmed, or moved 'stepwise' by completing user-defined vector displacements. These vectors can also be manipulated via menu-driven arithmetic operations: for example, they can be multiplied, added, etc.

Synthetically, the kinds of manipulations of representations afforded by the six DDAs can be grouped into two categories, that of expression creating, editing and executing and that of dynamic manipulation of a mathematical object. There is a great diversity in the ways these two kinds of manipulation co-exist and interact in each of the DDAs. Aplusix and Cuislet place emphasis on the former category for very different reasons. In Aplusix, the point of writing and editing expressions is to observe and think about the feedback and to juxtapose the two available representations. In Cruislet, writing code or giving inputs to location values is done in order to generate avatar displacements. Casyopee and MoPix do afford dynamic manipulation but place it in a role which seems less primary than the remaining two DDAs. Manipulation of function graphs and geometrical objects defined by mathematical expressions are part of the semiotic activity afforded to the user. However, definition of expressions and equations and observation of the resulting graphs are also critical. In MoPiX DM is less important. It allows the setting up of the objects location and shape before the running of a simulation, the manipulation however for meaning making is the writing of equations and the observation of their effect on the simulation under construction. Alnuset and MaLT on the other hand place dynamic manipulation in a critical role and in impressively different ways. In Alnuset, the connections made by the users amongst the three representations by means of dynamic manipulation of the innovative number line are at the heart of the designers' Didactical Functionality agenda. In MaLT, the reciprocal use of the variation tools and editing of the programming code is what matters. Dynamic manipulation in this case provides a feel for the essence of program definitions and parameters. In that sense, with respect to all 6 DDAs it is quite hard to capture what is the essence in manipulating the representations with respect to meaning making. The importance is not what is manipulable, but rather by which process does manipulation amongst other semiotic activity support mathematical meaning making. This issue has been widely addressed in the cross-experimentation.

Relationship to standard systems of representation, "innovation/acceptance"

Aplusix

Aplusix is well connected to established 'usual' systems of representing algebraic expressions and equations. The tree representation is flexible with regard to the "nodes" it can take, which maps to many levels of "usual" representations. The spatiality of the tree representation is entirely managed by the system. It may connect to geometrical and/or graph-theoretic mathematics. It also connects well to mathematical logic and to computational theory of expression parsing.

Casyopée

There are very clear connections between 'usual' functional representations, innovative ways of clarifying the role of domain and range in functional definitions and of using the dynamic aspects of computers to emphasize the role of parameters in function definitions. The connection with geometrical dependencies is consistent with an ontological view of the notion of function.

Alnuset

Symbolic/algebraic expressions, equations/inequalities, and propositions are standard, as are the Cartesian graphs of those objects. The number line representation is of course ubiquitous, but the "high ceiling" enabling complex operations and combinations of operations, along with its dynamicism, makes this number line quite innovative.

MoPiX

Functions are defined usually in terms of t, and match traditional parametric representations of functions. But these functions serve the purpose of animating and otherwise controlling very novel technological objects – in a way, reversing the “modeling” role of applied mathematics.

MaLT

Representations used in this DDA are innovative. Even in the case of Cartesian graphs, students construct their own graphs that control objects in the environment in novel ways.

Cruislet

The presentation of vectors as a mode of navigation in a geographic environment, programming aspects, and the notion of functional relationships between avatars that can be discovered is novel, but the use of polar and Cartesian coordinate systems, and enabling users to choose among these systems, has many applications in the standard curriculum.

In a sense, the six DDAs were ordered with respect to the distance of the available representations to the standard mathematical ones. But this issue is not so simple. First of all, there is the issue of the distance of the mathematics in each DDA to the mathematics traditionally taught in school and to the traditional epistemologies of school mathematics. In Aplusix for example it is quite clear that the designers’ choice was to minimize the distance in all those respects approaching the tasks of factorization and juxtaposition of expressions from standard to tree in the usual ways in which they are perceived in today’s schools. In MoPix, although representations of expressions are relatively close to standard styles, the epistemology is at a distance to many curricula by placing mathematics in the role of a conceptual tool for engineering. The same goes for Cruislet where mathematics is a means for navigation in space. Casyopee and Alnuset use standard representations but contain one outlier representation to allow for challenges to epistemological approaches, i.e. dynamic manipulation and connections between algebra and geometry in the former and dynamic manipulation of a number line in the latter. A second complexity to a linear ordering of the DDAs is the issue of the process of meaning making activity and the distance of that to the standard classroom activity. Can Aplusix which is tuned to enter present classroom life be used to challenge the ways in which students do mathematics? Can Cruislet which contains distant representations and challenging epistemological profile be incorporated in some kind of activity of today’s classroom? In retrospect, this notion of distance is not only more complex but also critical to understand how the DDA was designed and how it can be used across contexts.

Theoretical visions of representations, and of digital artefacts.

Aplusix

The designers suggest that Aplusix is intended to emphasize the difference between what the DDA framework report refers to as “procedural” and “structural” aspects of algebraic equations, expressions, and inequalities. This goal is achieved by providing a number of tools and tasks to students in which these algebraic objects are converted to/from 'usual' representations to tree representations: by highlighting and having the system compute different components of those representations, or by decomposing those representations into their constituent operands (which are featured at the 'bottom' or 'structural' level of the tree) and operators (which unite operands into higher-level branches of the tree). When a students' answer of a conversion task is compared to the expected answer, it is judged correct if and only the underlying expressions are identical, e.g., a tree representing 2+x is not accepted with regards to a tree representing x+2. Accepting the answer modulo commutativity has been by the designers as a question requiring more reflection.

Casyopée

Activities about functions are classified in a grid. The rows correspond to different level of objects represented. (1) Covariation and dependency in a physical system, (2) Covariation and dependency between magnitudes or measures, (3) Mathematical Functions of one real variable. The columns list different kind of representations of functions adapted from Tall (1996) and types of algebraic activities adapted from Kieran (2004). This grid helps to see how Casyopée's functionalities help to connect activities on different representations and levels of objects.

Alnuset

In this DDA, the digital artifacts are the equations themselves, which serve as constructions assembled from component parameters, operations, and manipulations.

MoPiX

In this system, mathematical expressions and equations are a means to program systems of objects to behave in desired ways. Rather than representing mathematics, mathematics are used to represent desired visual behaviors. Digital artifacts can then be saved and shared via a third XML-based representation appropriate for web dissemination.

MaLT

As in MoPiX, mathematical expressions in MaLT are a means to construct geometrical figures. In MaLT however, users are expected to gradually perceive of both figures and expressions as mathematical objects. In MaLT Digital artifacts are generic constructions that can be instantiated, but dynamically manipulated, and within which mathematical relationships of interest are embedded and extracted via direct, open-ended manipulation.

Cruislet

The balance between expressions as objects and as means for constructions is similar to MaLT even though the terrain for construction is drastically richer in information. The construction of meaningful behaviors – navigations through a familiar geographical environment – as performed through programming stepwise mathematical artifacts – vector displacements – is novel

Synthetically, the six DDAs can be organized in two broad groups. The former three were designed mainly within a framework combining TSD and ATD and also drawing upon IA. The latter three were designed with a constructionist perspective in mind, MoPiX in particular adopting a semiotic mediation framework in the analysis of use. This broad picture however captures very little of the design principles and the kinds of framework connections which can be used to analyze the use of the DDAs.

It cannot be clarified for instance how the juxtaposition of structural and procedural views of expressions may help students to generate meanings around these and what meanings different students may construct in a real classroom. Another example is Cruislet which is also based on a construct from activity theory, that of mathematization. What meanings can be generated throught the process of gradually discriminating which part of the semiotic activity mathematical and what traditional mathematics it can be connected to? The theoretical perspectives used for meaning making through dynamic manipulation also seem relatively obtuse.

Granted that dynamic manipulation may allow students to understand some mathematical connection mainly as a phenomenon but how can this semiotic activity relate to the construction of mathematical meanings such as dissociation from the tangible instances of graphics and proof? Which framework can show us the extent to which what is considered as shared amongst the mathematics education community works, i.e. that immersion in a more dense reification may lead to mathematical abstraction?

In that sense, rather than each DDA designer simply clarifying the ways in which the respective DDA was built with a general theoretical framework in mind, we need a more precise language taking into account the epistemology and the knowledge of the ways in which semiotic activity may generate mathematical meanings. For example both MaLT and MoPiX are constructionist DDAs and users construct simulations and geometrical objects respectively. The epistemology however is different in MoPiX there is a kind of phenomenology engulfing mathematical activity while in MaLT the extent to which the constructed objects will be considered as tangible or representations of ideal objects is open.

Finally the question of theoretical framework of design and theoretical framework of analysis of use has not been addressed analytically. Can the use of MaLT be analyzed by means of the semiotic mediation theory or the TDS theory through the application of a classroom scenario? If yes, what connections need to be made to its constructionist nature?

The profiles confirm the observation already made in the ReMath work that, while each team of designers is more sensible to one or another framework (TSD, ATD, AT, constructionism…), the design decisions, especially regarding representation and manipulation, also depend on an epistemological view of the notions, as well as on pragmatic considerations. The specific theoretical background of each design team is certainly visible at the surface level of its own DDA, but it was not an obstacle to cross experiments, showing that, more deeply, the teams share common epistemological views regarding the mathematical objects represented, or at least, understand each other at this epistemological level.

If not developed with Web "2.0" in mind, how might it be accommodated?

Aplusix

Given the specific, guided nature of tasks within this system, it seems that the best Web 2.0 application of this DDA is for assessment rather than the sharing/distribution of mathematical artifacts.

Casyopée

Casyopée encourages reflection on solving processes, by facilitating writing of reports. A html based notepad is offered with facilities for recording actions and storing symbolic results, graphs, figures… Web "2.0" features could be implemented to favor collective writing of reports and sharing intermediate solutions.

Alnuset

It would be interesting to enable students to share and compare one another's' solutions to generic problems in the form of equations and expressions constructed in Alnuset to see how equality (in specific cases), equivalence (in generic cases), and other relationships between different constructions of algebraic objects can be obtained.

MoPiX

Fully compatible with "Web 2.0"-style delivery: animations can be saved using MathML, for example, and produced animations can be shared and edited by others.

MaLT

As with Casyopee, the interface of MaLT is very involved, and requires a great deal of mouse interaction with objects, dialog boxes, and menus. It might be most appropriate for users to create and share the physical objects created within the system in an online applet format, which could be manipulable via limited interactions with a subset of the variational tools available in the current system, and as available for download for others to import into their own copy of the software.

Cruislet

Cruislet is based on a GIS system which is compatible with Web 2 design. Even though it was not designed specifically for this, one can imagine a stripped-down version in which multiple students fly patterns on a shared map, and those patterns can be shared and perhaps downloaded for 'full featured' exploration.

The six DDAs can be broadly organized in two groups, those where Web 2 perspectives have or could obviously be used and those which need further development both in design and functionality terms. In the first group lie MoPiX, Aplusix and Cruislet all of which are based on web 2 designs of distributable uses over the web, yet in different ways. The other three have rich interfaces and functionalities and need further design adjustments to make didactical use of their operation on the web. In all cases however, the idea of share-ability and discussion during the process of use has been proposed. There is a definite scope for such DDAs to be developed further in order to integrate the potential of web 2 designs and social tool functionalities. While this task cannot be considered as a trivial extension, it will be facilitated by the focus put in the ReMath project on social aspects of learning.

Addressing the DDA profile

It's clear from the analyses conducted above that the Remath project has created a diverse set of DDAs that address a variety of mathematical topics using quite distinct representations. Although the DDAs were created with quite different theoretical frameworks, the DDA profile framework captures the important ideas, features, and pedagogical uses of the DDAs.

A key characteristic brought to the surface by means of the profile was the detail on exactly what interactions and manipulations are considered the important objects/concepts in the system. These details are crucial to understanding the real affordances of the DDAs in context. Apart from a description of some of the HCI aspects of the software it was particularly important to capture how the user is expected to experience the DDA interface. The particular reference to the mathematics, the representations and the kinds of expected uses of these was important for clarifying the designers' choices and the DDAs affordances. The analysis of representations and their uses in terms of traditional representations provide a point of reference illuminating the kind and essence of each representation in a relatively de-contextualized way. The 'activities' section captured the complexity of how the user is expected to work with the software and how mathematical meanings could be constructed. The section on pedagogical/intervention agenda gave some of the background driving the designers choices and expected uses of the DDA in educational settings e.g. along the continuum of overt challenge or to seamless support of traditional classroom activity.

In an attempt to see how the profile might work for a DDA outside the ReMath project and the European scene, the teams asked Uri Wilensky to write up a profile of NETLOGO, an established yet with forerunning ideas DDA used by a large community in the States. Applying the profile framework to the quite different NetLogo software successfully captured much of the affordances of the NetLogo DDA (see Appendix 1).

V. Building on the cross-experimentations: the cross-case analyses

In part III, we have recalled the main elements of the experimental process developed in ReMath and presented in a synthetic way its first outcomes as reported in the deliverable D14. Interesting similarities and differences have been pointed out in the way the common research question (CRQ) was rephrased by the different teams, in the ways the learning potential of the new representational affordances offered by the DDAs was approached, and consequently in the scenarios prepared and the associated expectations. Intriguing differences were also observed between the familiar and alien experimentations for some teams. What can be the exact sources of such similarities and differences? Up to what point do they reflect differences or similarities in theoretical approaches? Contexts? Up to what point do they reflect the diversity of the DDAs themselves described above? How to benefit from this experience for improving the initial version of the ITF? For answering these questions, the decision was taken to complement the local analyses made by each team by cross-case analysis. With these cross-case analyses which systematically compare the two experiments involving the same

DDA[3], our aim is to pave the way towards the type of ITF evoked at the end of the deliverable D9, progressing in terms of networking from a state of increased understanding, comparison and contrast towards a state of coordination and partial integration. In this part, we first present the methodology we have used for developing these cross-cases analyses, and then synthesize their main results. The details of the analyses can be found in an appendix to this deliverable.

V.1. Methodology

The idea of cross-case analysis starts from the hypothesis that the systematic comparison of the two experiments involving the same DDA can help us understand the reasons for the similarities and differences generally observed, and make us able to better interpret these when appropriate in terms of proximity or distance between theoretical frames. It is also hypothesized that it can make emerge interesting connections and complementarities between theoretical frames that local analyses done within a particular culture and context cannot reveal.

We consider also important to link this cross analysis to the ITF structure as much as possible, paying attention to both theoretical frameworks and contexts, and organizing the analysis around the three dimensions of didactical functionalities, using the language of concerns when appropriate.

Finally, we are aware that, in order to make sense for a larger audience, the theoretical elaborations resulting from such cross-analyses need to be associated with illustrative examples. The common frame fixed for developing the cross-case analyses and presenting these in a way favoring their exploitation for the final version of the ITF results from these considerations. The comments attached to each item of the structure, from the second one, reflect hypotheses made in the light of the analysis of the cross-experimentation already made regarding factors which have certainly contributed to observed similarities and differences, and beyond theoretical approaches:

- Differences between the ReMath DDAs, regarding the kind of representations they offer and the semiotic creativity these make possible;
- Differences in terms of math educational (didactic) knowledge potentially supporting the design, depending on the domains at stake (algebra/ 3D geometry);
- Differences in the relationship with a given DDA created by the familiar /alien position;
- Differences in the status given to the experimentation and the DDA by the different groups of teachers involved in the experiments.

The cross-case study structure is thus the following:

I. Case identification: teams involved, DDA considered

[3] In the case of 5 out of 6 DDAs two different teams experimented peadgogical plans involving the use of a given DDA. One DDA was experimented by three different teams.

II. Contextual elements:

- information concerning the school level, the number of students involved, the number of hours, the classroom organization and type of sessions and their situation with respect to the ordinary life of the classroom;
- information about the relationships between the research team, the teachers and the schools involved in the experiments, the negotiations that took place between the schools and the research team for this experimentation if any, and how research interventions are more globally perceived by the educational system.

III. Theoretical frames

Identification of the main theoretical frameworks used by the two teams with a distinction between general theoretical frames and frames related to the specific mathematical domain concerned by the experimentation. As theoretical frameworks can have very different sizes, it is asked to make the constructs of the theories mainly used as precise as possible. Moreover, if there are differences between the theoretical frames used in the design of the experimentation, the analysis of it and/or the cross-analysis for a given team, these should be mentioned together with the rationale for change.

IV. Comparison of didactical functionalities

Presentation of the current state of the analysis of didactical functionalities made by the two teams regarding their experimentation, in a way allowing a clear vision of similarities and differences.

V. Results of the cross-case analysis together with illustrative examples

Synthetic presentation of the methodology of the cross-case analysis, focus on its most interesting results, having in mind the ITF, that is to say the look for similarities and differences, complementarities and possible connections, and their impact on the design of the experimentation, its implementation or the analysis of its results. The presentation has to make clear the respective influence on the discussed phenomena of theoretical elements (both proximity and distance), of contextual elements and of characteristics of the DDA. The points mentioned should be illustrated by specific examples presented in a way that makes it possible to use them in the theoretical landscape with hyperlinks.

VI. Potential offered for the theoretical landscape

A synthetic view of what is offered by this cross-case analysis for contributing to the elaboration of the theoretical landscape, including the identification of interesting boundary objects with precise examples of the potential they offer, suggestions for contribution to the (STF) Shared Theoretical Frame or illustrations contributing to making sense of it, as well as suggestions regarding the improvement of the language of concerns.

V.2 Synthesis of the main results

Aplusix cross-case analysis

Aplusix has been experimented by three different teams with students ranging from grade 7 to grade 10. The length of the experimentations ranges from 5,5h to 18h, with significant differences between the French and Italian experimentations. Regarding the theoretical frames, an interesting characteristic of this cross-case study is the common reference of the three teams to the notion of semiotic register of representation due to Duval. Three semiotic registers are identified: standard algebraic register, tree register and natural language. At a more macro-level the three teams differ in their theoretical approaches, MeTAH team referring to the Anthropological Theory of Didactics (ATD) and organizing its scenario in accordance with the 6 study moments identified in ATD while the two Italian teams refer respectively to Activity Theory (AT), and the Theory of Semiotic Mediations (TSM). In fact, TSM is also based on AT, but it colors differently the design, making the designers focus more systematically on the process of production and evolution of signs through the use of the artefact, and to the role of the teacher in its effectiveness. What is also interesting is that these differences influence the way the feedback of Aplusix is considered and used.

Similarities between the three teams are also evident in the identification of an educational goal. These similarities can be traced in shared knowledge regarding the didactic of algebra, and the shared acknowledgment of the importance to be given to the notion of algebraic equivalence, and to the development of a structural view of algebraic expressions this allows. As could be anticipated, MeTAH for which Aplusix is the familiar DDA chooses to test the potential provided by the new register of representation created in ReMath: the tree representation for making students distinguish between procedural to structural aspects of algebraic expressions. The possibility offered by Aplusix to use either controlled tree mode or free tree mode is used as a didactic variable in the scenario for organizing the progression along the study moments as carefully explained in the cross-case analysis. The UNISI team referring to TSM builds a scenario which aims at making the interpretation of feedback-signs for equivalence or non-equivalence evolve, through the tasks proposed and thanks to the mediation of the teacher, from a primary and common sense interpretation towards a developed and mathematically meaningful interpretation. This is performed first in the standard algebraic register, then through the connection between this register and the tree register. UNISI team insists on the fact that the link between feedback and their mathematical meaning is far from being automatic and that its development needs to be carefully orchestrated by the teacher through the design of appropriate tasks and collective discussions. The analysis provides evidence for that, tracing in excerpts of protocols semiotic chains attesting the progressive development of a mathematical sign attached to the notion of equivalence, and the role of the teacher in it. The ITD team, for its part, referring to activity theory uses Aplusix feedback for making visible contradictions between paper-pencil and Aplusix solutions to algebraic tasks, and for fueling classroom discussions. The tasks proposed to the students are first tasks of conversion between standard and tree representations, and the experimentation shows that the two conversions do not raise the same cognitive difficulties. It is interesting to notice that the importance given to such conversion tasks, also present in the other scenarios is coherent with the reference to Duval's theory. But

Aplusix seems in this particular experiment limited to a status of verification tool. A priori, it seems difficult to interpret this characteristic of the experimentation as a mere consequence of the theoretical choice of this team even if the cross-case analysis explicitly claims that the tasks follow the AT model.

This cross-case analysis provides thus interesting insights for our purpose. First it shows three different coherent uses of the theory of semiotic registers attached to the respective connection of this local theory with three more global theories, and the influence also played in the three designs by the accumulated knowledge in the didactic of algebra regarding procedural and structural aspects of algebraic expressions and the central role given to the equivalence between expressions. Thanks to this shared background, the tasks built by the three teams, while different, make sense for the three approaches in our opinion. Aplusix feedback can be seen as a boundary object for analyzing similarities and differences between the three approaches. The three designs indeed pay particular attention to this feedback, but under the umbrella of ATD these are used to foster the development of a suitable technique in Aplusix environment, under the umbrella of TSM they are seen as signs whose interpretation must progressively be charged from mathematical meaning, under the umbrella of AT, they are asked to reveal contradictions and orientate students' activity towards the solving of these contradictions. These visions of feedback are not contradictory, rather they show interesting complementarities between theoretical perspectives, enriching the cognitive levers highlighted by each of these separately. Another point is the differentiation in the role given to the teacher in the three designs, apparently more emphasized in AT and TSM. Regarding contexts, they have obviously influenced the length of the different experimentations, and the reduced length of the MeTAH experimentation, concerning its familiar DDA, can be interpreted as a sign of the strength of institutional constraints in France.

Alnuset cross-analysis

Alnuset cross-case analysis, when compared to the previous one, focuses only on a limited part of the two respective experiments, those where the didactical functionalities identified by the two teams involved: ITD and MeTAH are the most compatible. For ITD, it corresponds to a very limited part (1h40) of the whole experimentation (20h), while for MeTAH it is about one half of the experimentation. The difference between the global lengths of the two experimentations seems to confirm the differences of institutional pressure between France and Italy already pointed out in Aplusix cross-case analysis. This difference in contexts is also visible in the way the experimentations are dealt with. ITD makes clear that what is proposed to the student represents an important curricular change, but this change seems rather easily negotiated with both the teacher and the institution. MeTAH, for its part, tries to define its short experimentation in order to cope as much as possible with the existing curriculum.

This observed difference is certainly related to contextual characteristics, but it also reflects differences in the theoretical frameworks at stake. As was the case in the Aplusix experiment, the two teams share the same didactic background as far as algebra is concerned. The nature of Alnuset led them to pay particular attention to the notion of algebraic equality and the well known conceptual changes required regarding equality when passing from numbers to algebra. The two teams share the hypothesis that the operative and representative opportunities of Alnuset can be effectively used to mediate these conceptual changes. As was also the case with Aplusix, at a more general level the two teams refer to different theoretical frames: AT and ATD. It is interesting to notice that the description made by ITD of AT is much more detailed than was the case for the Aplusix cross-case analysis. Explicit reference is made to the Cole's and Engeström's model. It leads to see in Aplusix a tool for making the algebraic rules, which are first individual-community mediators, become objects of learning. According to ITD, such a shift of focus requires some breakdown, and this explains the importance given in the scenario designed to tasks generating contradictions and breakdowns in the students' activity. As was the case with Aplusix, the comparison of paper and pencil solutions and Alnuset solutions is considered a privileged source of such contradictions. Another interesting point emerges from this analysis. ITD team claims that breakdowns and shifts of focus are mainly of semiotic nature, and moves for making sense of these from the Duval's approach in terms of semiotic registers of representations to Pierce's triadic theory of signs, making particular use of the distinction between three kinds of signs: indices, icons and symbols introduced in this theory. The indexical and iconic potential of Alnuset signs for representing conditioned equality, relationship of equivalence and relationship of equivalence with restrictions, and for representing the structure of expressions is pointed out, and it is inferred from it that Alnuset can mediate the development of the control over the conditions that determine the equality or equivalence between expressions (through the Algebraic line), and of the way to use the rules of algebraic transformation (through the Algebraic manipulator).

MeTAH team for its part, goes on relying on Duval's theory and ATD. It is interesting to notice that the use of Duval's theory leads to some negative conclusions. Considering the three types of semiotic activities identified in the theory: formation of a representation in a given register, treatment inside a register and conversion between two registers, MeTAH researchers argue that, due to the automaticity of most conversions in Alnuset, and the limited possibilities of treatment offered outside the Algebraic manipulator that would require too much time for instrumentalization, they mainly see an exploitable potential in the links between the algebraic and graphical registers mediated by the dynamical graphical register. Moreover, they insist on the relevance of establishing such links for the grade 10 curriculum corresponding to the level of their experimentation. There is no doubt that this curricular proximity influences their experimental choice, under the contextual constraints met. The second theoretical framework that MeTAH uses is ATD. The use is slightly different from that made in the Aplusix case, and it partially results in our opinion from the curricular proximity just evoked. In the first experiment, the accent was put on the different “moments of study” with the meaning given to this notion in ATD, this time it is put on the comparison between paper and pencil and Alnuset praxeologies for three types of tasks viable in the two environments: determining if two expressions are equivalent, solving equations of the form $f(x)=0$ and $f(x)=g(x)$, while focusing on the technique part of praxeologies.

The analysis made by ITD team illustrates the way index and iconic signs help students make sense of equivalent expressions and equations in the Alnuset environment. The analysis made by MeTAH team illustrates how the solving of appropriate tasks can make students evolve from a perception of Alnuset as a verification tool to a perception of it as a tool for exploring and conjecturing, showing that this second use seems more productive from a mathematical point of view. The analysis also evidences the limits of Alnuset instrumentalization in such a short time.

It is thus interesting to notice that, even if the two teams share the same epistemological and didactical background in algebra, leading them to globally identify the same interesting characteristics of the DDA and fix close educational goals in their selection of didactical functionalities, the intertwining of contextual and theoretical differences leads them to different modalities of use and focus in their respective analysis. We also observe an evident coherence between the experimental scenario elaborated by ITD for which Alnuset is the familiar DDA and the epistemological profile, while for MeTAH referring to Duval's theory, it is more difficult to make use in a controlled way of the semiotic potential of this DDA which escapes Duval's categories and is aligned with Pierce's theory of signs.

Casyopée cross-case analysis

With the Casyopée cross-case analysis, once more a French and an Italian teams are involved: DIDIREM and UNISI. The software: Casyopée is a DDA organized around the notion of function, and offers different representational tools for approaching functions and the functional modeling of geometrical situations. Despite the fact that the experimentations take place in two different countries with different educational cultures, the contexts offer some similarities: experimentation is carried out with high school students at grade 11 (France) and 12 (Italy); the duration of the experiments is around 10 hours; the experiments are carried out with experienced teachers used to collaborate with the researchers in ordinary conditions, if one excepts the fact that they take place, at least partly, in a computer laboratory.

The two teams differ nevertheless in their theoretical perspectives. As already explained in the synthesis of Aplusix cross-case analysis, UNISI team mainly relies on the theory of semiotic mediations (TSM) while DIDIREM team is inspired by the Instrumental approach (IA), the Theory of didactic situations (TDS) and the anthropological theory of didactics (ATD). Beyond these differences, the two teams share a common kernel of didactic knowledge already stabilized around functions and dynamic geometry, attach similar importance to connections between semiotic registers and settings (algebra and geometry mainly here) with the meaning given to this term by Douady (1986). These similarities and differences impact both the selected didactical functionalities and the analysis of the experimentation. The two teams pay particular attention to the same characteristics of the DDA: the specific features of the DGS environment of Casyopée allowing the user to create points with parametric coordinates, the new “geometric calculation” environment situated at the interface between algebra and geometry, and the possibility it offers to the students to try easily to associate different functions to the same geometrical situation, and compare the pertinence of these choices for creating functions and solving the problem at stake, the commands for creating and manipulating parameters, beyond more standard characteristics of this DDA. The two teams are also sensitive to the possible feedback resulting from the interaction with the DDA. With respect to educational goals, the two pedagogical plans deal with functions seen as co-variations, variables and parameters. Some mathematical tasks proposed to students are very close, and one similar. Beyond these similarities, the cross-analysis also shows the existence of differences. UNISI pedagogical plan is designed for older students supposed to be already familiar with functions. It aims at mediating and weaving meanings related to the notions of function, variable and parameter for reaching a deeper level of consciousness of these mathematical notions, and their re-appropriation in the context of modeling, up to a shared and decontextualized formalization. DIDIREM pedagogical plan aims at the introduction of new knowledge playing on the cognitive potential offered by the diversity of Casyopée semiotic registers and settings. The respective didactical functionalities also differ in terms of modalities of use. The UNISI pedagogical plan has an iterative structure which alternates activities with Casyopée and class discussions carefully orchestrated by the teacher. Students are required to regularly produce written reports. The DIDIREM pedagogical plan seems especially attentive to the intertwined progression of mathematical and instrumental knowledge. Moreover tasks are designed in order to a priori make possible students' autonomous work. Obviously, the theoretical differences mentioned above partially explain these differences. The important role given by UNISI team to classroom discussions is fully in line with TSM for which the teacher plays a central role for fostering the evolution of students' personal meanings towards those targeted. The reference made by DIDIREM team to IA explains the progressive instrumentalization of the artefact along the sessions which contrasts with the initial familiarization session organized by UNISI team. The accent put on the potential for students' autonomous work in the design of tasks can be traced in the TDS and the importance this theory gives to adidactical interactions. But, as pointed out above, these differences do not prevent the two teams from building very close mathematical tasks for this experimentation and even share some of these. According to us, these similarities and the ways they express through the careful design of tasks reflect the importance given by the two teams to the epistemological concern.

Regarding the analysis of the experimentation, the two teams decided to focus the cross-case analysis on collective phases. The reason behind this choice was the fact that in the UNISI experimentation, a key element of the semiotic mediation process was made from the collective discussions. In the DIDIREM experimentation, the most similar ingredient consisted in institutionalization phases, but in these the interplay of responsibilities between students and teacher as well as the role given to representations seemed different. Thus, understanding better the differences between these two ways of passing from the individual or group work to the collective, and its effects in terms of representations was considered as an interesting approach towards the identification of potential complementarities between the theoretical approaches of the two teams. The transcript of a collective discussion for UNISI team and that of a laboratory session where there was evidence of collective interaction between students and the teacher for DIDIREM team were thus selected and the two teams tried to develop a combined analysis of these. The cross-case analysis report shows this work and its outcomes through the consideration of selected short episodes. Interesting differences and possible complementarities are evidenced through this process. For instance the cross-experimentation has proved that the theoretical frameworks used by DIDIREM provide this team with very useful constructs for developing tasks highly significant from an epistemological point of view, and for analyzing their potential for mathematical learning through the analysis of the milieu, of the possible adidactic interactions between the students and this milieu, paying particular attention to the feedback resulting from this interaction. The selected episodes make clear that nevertheless the tasks that have been designed and implemented by DIDIREM correspond mainly to adidadic situations of action. In other terms, strong attention has been given in the design to situations of action, much less to what is called in TDS to situations of formulation and validation, which are asked to ensure a progression in the functionality of mathematics knowledge from a tool for productive action to a tool for efficient expression and communication, and then a tool for proving. Of course, as pointed out in the analysis, sub-tasks 2 and 3 ask students to situate in a reflective posture with respect to action and enter in a formulation process, but they do not obey the requirements imposed to a situation of formulation and their devolution is not organized. This is not at all an unusual phenomenon but has evident consequences on the way semiotic mediations are approached as made clear by the cross-case analysis. This analysis shows classroom dynamics where situations of action are directly followed by institutionalization phases where the teacher tries to make collectively emerge from students' adidactic experience the mathematical knowledge aimed at. The condensation of formulation, validation and institutionalization processes that this dynamics induces is an evident source of difficulties which manifest for instance by many Topaze effects where a question posed to students is progressively rephrased up to automatically generate the expected answer. These difficulties in fact help us understand the complexity of the semiotic processes at stake, in the transition from successful activity with the DDA and for the solving of mathematical tasks towards the articulation of the mathematical knowledge implicitly engaged in action or potentially emerging from it. This becomes especially difficult if students are very quickly imposed linguistic forms taking evident distance from the language of action even if they do not constitute fully decontextualized language. The analysis thus shows the importance of having theoretical constructs for supporting the sensitivity to these semiotic processes, the respective roles that teachers and students can play in these and forms of didactic design able to support them. There is no doubt that current refinement of TDS such as those developed by Hersant and Perrin-Glorian (2005) for analyzing what they call "interactive synthesis" or those developed by Sensevy & al. (2005) in terms of co-construction of knowledge by students and teachers are helpful for approaching this theoretical needs. In the cross-case analysis, these tools are provided by the TSM constructs used by UNISI team. Both the selected episodes from the classroom discussion and the analysis provided by UNISI team of the collective part of the laboratory session show the interest of these tools and their possible complementarity with DIDIREM perspectives. For instance, the analysis of collective discussions show that their goals are made clear to students from the beginning. The identification of semiotic chains and the analysis of the way both teacher and students co-produce these illustrate powerful levers for co-construction of knowledge in the classroom according to Sensevy & al (2005). It is interesting also to point out that a superficial vision of interactions between teacher and students in collective discussion could lead to interpret these in terms of Topaze effects while a deeper analysis developed shows that the dynamics they obey is different from that characteristic of Topaze effect and leads to strongly question inferences that could be drawn from student' correct answer when eventually this is produced. The cross-analysis carried out more generally points out the importance to be given to phases where students are asked to rethink about their actual use of the artefact for solving the task. As claimed, it is a "necessary" step for recalling the personal experience developed through that activity and may be useful to focalize on the actual use of the artefact. Moreover the request of a verbal production is likely to lead one to a first de-contextualization expressed by written signs that provide a first elaboration of meanings.

Another point we would like to mention is the fact that this cross-case analysis reveals a different use of the ideas of Rabardel regarding instruments and instrumental genesis, even if both teams are equally sensitive to instrumental issues and share the reference to his research work. The differences observed affect both the design and its implementation, and are especially visible in the two first illustrative examples. The collective discussion takes place without access to the DDA and this is a deliberate choice. It aims at supporting a double de-contextualization process concerning both the task and the means for solving it. Students are asked to reflect on what they can draw from this experience for solving a class of tasks with and without the DDA. The fact of having the collective discussion in a place where Casyopée is not available is expected to facilitate the de-contextualization process and the semiotic game, making impossible to replay action, obliging to evoke it through discourse and gestures. The artefact is seen as a tool for semiotic mediation and what is learnt has to resist to its absence. The vision underlying the instrumental approach used by DIDIREM which has been developed in the context of CAS is not exactly the same. According to this vision, mathematical knowledge resulting from instrumental genesis deeply intertwines mathematical and instrumental ingredients and coherently with this vision they do not include in their didactic plan any attempt at separating these. De-contextualization focuses thus only on the transition from one particular task to the more general class it belongs to. The fact that the instrumental approach has been developed in the context of a long term use of CAS has certainly contributed to this theoretical position.

In its last part, the cross-case analysis of Casyopée proposes another complementarity between the two perspectives in terms of didactic contract, relying also on the doctoral thesis by Falcade, out of the ReMath project. We do not synthesize this part here but will take it into account in the last part of this deliverable.

Mopix cross-case analysis

With MoPiX cross-case analysis, we enter in a case quite different from those presented above. First because of the characteristics of MoPiX, a DDA more distant than the previous ones from usual artefacts used in mathematics classrooms, second because this DDA situates at the interface of mathematics and mechanics in terms of domain, which leads to an experimentation in engineering courses, third because MoPiX design relies on a constructionist perspective in line both with the theoretical approaches of the familiar and alien team. We thus anticipate a convergent analysis and smooth connections between the two experiments, and also complementary insights to those provided by the previous three cross-case studies.

MoPiX cross-case analysis is consistent with these expectations, but it also shows despite evident commonalities, differences in contexts and theoretical frameworks and the effect of these. Differences in contexts are evident. The IOE team works under strong institutional constraints. The pedagogical plan has to show its compliance with the defined curriculum and students have to be convinced of the interest of the experimentation for the preparation of university entrance examinations. Sessions cannot take place inside the normal schedule. Conversely, ETL succeeds in negotiating access to the vocational institution and in designing a programme of work independent of specific goals in the standard curriculum for 25h. The institutional situation of the domain dealt with by MoPiX varies also from one country to the other: mechanics is part of the standard mathematics curriculum in England while separated from it in Greece. Despite these differences, what is proposed in the two experimentations is quite distant from usual practices.

Regarding theoretical frames, the primary theoretical framework adopted by the IOE team is multimodal social semiotics highlighting the different potential for meaning offered by different modes of communication. As clearly expressed in the cross-case analysis, social semiotics is not a theory of learning, and is not sufficient by itself to inform the design of activities for learning. This is the reason why the design of MoPiX and the pedagogical plan are also influenced by a broad constructionist theoretical frame. However, interaction with physical representations is not considered by itself sufficient for effective learning. It is supposed that learners need to make sense of their experience of manipulating representations in the context of social interaction with peers and with teachers in order to be able to challenge and test alternative conceptualizations and forms of reasoning. Hence the importance attached in the IOE pedagogical plan to the support provided to social interactions. ETL team for its part situates within a constructionist perspective, incorporating in it a social sensitivity. From this perspective, MoPiX is thus seen by ETL as a learning environment giving students the opportunity to explore, manipulate and build animated models representing different phenomena and situations, and engaging in meaning making processes. For ETL team, the driving force behind any constructionist activity in MoPiX is the use of mathematical formalism. Through this mathematical formalism in terms of equations, students are considered to have a deep structural access to the mathematical models underpinning the behaviors animated on the screen. Another influence of this constructionist perspective is visible in the way ETL considers MoPiX as a potential source for half-baked microworlds whose instrumentalization by students will generate meaning making processes (Kynigos, 2007 a).

These contextual and theoretical characteristics reflect in the respective scenarios built for the IOE and ETL experimentations. For IOE team, the experiment has a clear curricular goal: the development of the concepts of velocity and acceleration. Within a general interest in students' use of the semiotic multimodality offered by MoPiX, the team focuses on the way students represent these concepts. For ETL team, the focus is on the construction of models using mathematics and the research interest concerns more specifically the potential offered by MoPiX equational formalism for meaning-making processes. The respective analysis provided by the two teams of episode 2 coming from ETL experiment makes these rather subtle differences quite visible. IOE team interprets the episode, tracing the semiotic chain which, through a diversity of semiotic systems leads from an everyday description of a visual phenomenon to be produced to a set of MoPiX equations, without inferring from that analysis cognitive conclusions. ETL analysis for its part focuses more on the links between MoPiX successive formal expressions, the role that existing equations play as templates for the construction of new ones adapted to the problem to be solved, and draws from that the existence of conceptual connections.

Considering the three first cross-case analysis, we also identify interesting differences. With MoPiX, the curricular distance between the DDA and standard educational tools substantially increases. But we observe that according to the context, the effects of this curricular distance on DDA acceptability may vary. Another difference relies in the respective influence taken in design by theoretical constructs or knowledge specific to the domain at stake, here mechanics, and more general constructs. ETL team does not mention any domain specific construct, which contrasts with the role played by notions such as that of structural equivalence in the previous cross-case analysis. We also would like to point out the vision of instrumentalization linked to the use of half-baked microworlds and constructionism, which seems rather different from that developed for instance in the Casyopée cross-case analysis where instrumental approach is more tightly connected to ATD. Finally we would like to point out the use of the notion of semiotic chain by IOE team offering evident similarity with that developed by UNISI team.

MaLT cross-case analysis

With MaLT cross-case analysis, we enter once more a new domain: angles and 3D geometry, but the two teams involved are the same as for MoPiX. We thus expect similarities with the previous case, but also some differences to be related to differences between the DDAs at stake or the local context of experimentation. There is indeed an evident difference in context with MoPiX experimentation: students are younger (aged 12-13) and thus in lower secondary school. Once more the ETL and IOE contexts are quite different. ETL experimentation takes place in a multicultural secondary school which has adopted the "Flexible zone" program. This aims at linking horizontally the content of all subjects taught by introducing a two-hour per week session during which students engage in cross-curricular projects and activities, where they are given the opportunity to collaborate with their peers in exploratory activities often based on the use of computers. This specific context allowed once more ETL team not to emphasize on closed didactical goals but to use MaLT in order to challenge students' construction of meaning for angles in 3D geometry along a rather long period: 18 sessions over 2 months. IOE experimentation does not take place in so favorable conditions. It takes place in a traditional school including a significant proportion of students coming from low-income areas. The pressure of national tests is high and the pedagogical plan is thus asked to match the official curriculum. Moreover the class chosen comprises low achievers who are not expected to do well in national test, feel marginalized and have often non-conformist behavior. They are not used to work in groups. The experimentation is well received, and the time allocated to 3D topics being short, two after school additional sessions and some arrangements are made in order to built a two weeks experiment including 9 sessions among which 4 with the DDA.

As could be expected, the two teams rely on the same main theoretical frame as in MoPiX experiment. ETL relies thus on a constructionist perspective leading the researchers to investigate how students construct mathematical ideas as situated abstractions through their exploration of and interaction with MaLT representations. Regarding the mathematical domain at stake, ETL team retains from the literature that students meet great difficulties at coordinating the different facets of the notion of angle involved in various physical contexts, a coordination which is supposed necessary for developing a mature abstract angle concept. In reference to the theory of conceptual field due to Vergnaud, ETL team also takes into consideration that the concept of angle does not make sense in 3D geometry in an isolated way but has to be considered as situated within a conceptual field including mathematical objects such as rotations. IOE for its part remains attached to multi-modal social semiotics, focusing on the multi-modal and multi-semiotic characteristics of learning environments, and the opportunities these give the students for making meanings with the representations available to them and choices about the most apt representations to employ in order to communicate their desired meanings. As ETL, IOE team recognizes angles and 3D geometry as difficult domains, and mentions for instance the difficulty to connect the characteristics of 2D representations of 3D objects that are often used for teaching 3D geometry and the properties of the objects themselves. It is thus considered that operating with different semiotic systems provides important opportunities for developing understanding of 3D objects. In reference to Duval, IOE team also considers that conversion between different semiotic systems, is an activity of fundamental importance for learning, and that MaLT is in that case helpful by the dependence it implements between semiotic systems facilitating conversions and associated abstractions of the properties of 3D shapes. This reference to Duval's approach was not present in MoPiX cross-case analysis, and in a similar way the theory of conceptual fields was not mentioned by ETL in that case.

These theoretical choices and contexts influence the identification of didactical functionalities and design. Both teams consider important to start from tasks often encountered in everyday physical angles situations such as revolving doors. Both teams pay particular attention to the new types of turtle turns offered by MaLT. These are related to the passage from one plane to another, and are asked to support the meaning making of angles as turns (seen both as actions and as the result of these), and through their measure. Beyond that, IOE challenged by the limitations of its experimentation and of the students' knowledge of angles focuses on the meaning making of Logo language and on its connection both with the representation in the turtle screen and with physical experience. ETL develops a more ambitious project using more systematically the different semiotic systems offered by MaLT for expressing and reflecting on the mathematical nature of angle as a dynamic spatial concept and challenge their conceptions of the conventions used to represent 3D objects in the computer screen.

As was the case with MoPiX analysis, even if some tasks proposed to the students are very close, the analysis provided of some interesting episodes from the two experimentations shows some subtle difference of focus in the analysis of the experiment between the two teams. ETL team explores the role of the body-syntonic metaphor in the students' use of the available representations in MaLT as well as its influence on the students' constructions of meanings for angle in 3D space. Logo programs, students discourse and gestures are thus used for instance in order to understand students' evident difficulties which are interpreted in terms of non differentiating between the world frame of reference considered as absolute and the relative vehicle frame of reference underlying the turtle move, and then trace the evolution of students' conceptualizations. IOE team for its part focuses on the extensive of gestures used by the teachers and researchers in order to support students' activity. They consider this set of gestures as a new semiotic system and analyze how students adopt these new signs and the relationships between the semiotic activities of the students and the teachers. These gestures are linked to the metaphor of "playing turtle" which seems inherited from the researchers' Logo culture in 2D space (shared by the two teams that similarly speak in terms of body-syntonicity). A very detailed analysis of gestures and their use in communication between students and researcher-teachers is thus carried out, leading to question the potential of the "playing turtle" metaphor in this 3D context, many of the movements required from the turtle being impossible for the human body within its normal environment. As was the case with MoPiX, no inferences are made of the analysis in terms of conceptualizations.

It is also interesting to point out that the two experimentations, and above all the IOE experimentation more limited and with more fragile students, make evident the challenge raised by the production of construction in the turtle screen and the interpretation of the 2D dynamic representations of 3D shapes that this DDA provides. The cross-analysis carried out show us that each team with the specific theoretical lens it uses helps us identify possible and complementary reasons at the difficulty of this challenge. These intertwine factors linked to the mathematical notions at stake, factors linked to the problems raised by the 2D representation of objects in 3D space, and factors resulting from the specific characteristics of the representations of objects and movements as implemented in MaLT when generalizing to 3D space turtle geometry. Within the instrumental approach not evoked here, the last ones would certainly be interpreted in terms of instrumental distance between paper-pencil and MaLT representations, and cost of instrumentalization. Within a theoretical framework paying specific attention to institutional constraints such as ATD, it could lead to question the viability of MaLT use in standard contexts. We get here the feeling that the constructionist perspective leads the researchers to see these difficulties more as a challenge, and for that reason we have used this term above, and to consider with more positive eyes the observations coming from this experimentation, tracing the subtle evolution of students' behaviors and trying to make sense of it in terms of conceptualization. This is of course all the more possible that the experimentation is a rather long term experimentation and that its ultimate educational goal can be let open, as is the case in the ETL context.

Cruislet cross-case analysis

Cruislet cross-case analysis with which we end this part of the report promises to be rather different from those synthesized above. First Cruislet is among the ReMath DDAs at the extremal distance with standard tools including digital tools of usual curriculum. Second the two teams involved in its experimentation: DIDIREM and ETL are also very distant in terms of theoretical perspectives. Moreover DIDIREM experimentation takes place in France, a country where education, as already pointed out, is under strong institutional pressure.

Cruislet cross-case analysis confirms these expectations. The influence of the context is immediately visible. In both countries, what is offered by Cruislet is very far from the curriculum but this does not have similar consequences on the experimentation. In Greece, a rather long experimentation is organized in grade 10 classrooms (20h in one class, 8h in the other one) without apparent difficulty while in France, making the use of Cruislet compatible with institutional constraints imposes for the experimentation to take place in very specific circumstances: multidisciplinary project work in grade 11 first, university workshops for grade 9 students then. Moreover, in the first case, the negotiation of the scenario with the teachers in charge of the experimentation is a rather complex process and, by the end, after the introductory sessions no group of students decides to choose Cruislet as a support for his multidisciplinary project.

From a theoretical point of view, ETL references are those already mentioned in the previous cross-case studies. They lead ETL team to consider this experimentation as the study of students' gradual mathematizations in a constructionist environment and as a path towards clarifying the idea of instrumentalization by design. This leads them to especially build on the characteristics of Cruislet as a half-baked microworld, and on the potential offered by the complex linkage of representations offered by this DDA for investigating the mathematical meanings that students construct regarding the notion of function as co-variation while navigating in 3D large scale space. For that purpose, students are asked to explore and decode black box Logo programs defining the relative displacements of two avatars, then encouraged to build their own rules, produce associated Logo procedures and thus create new such functional games. As this description shows, the continuity with the other experiments carried out by ETL is evident.

The situation is radically different for DIDIREM. The cross-case analysis shows the distance separating the two experiments carried out with Cruislet from the experiment carried out with the familiar DDA Cassyopée. From a theoretical point of view, the global references are the same: Instrumental Approach (IA), ATD and TDS. IA and ATD strongly influence the vision of Cruislet didactical functionalities and thus the design of the experiments. IA leads to pay particular attention to the instrumentalization needs of such a DDA, so complex and so far from those usually used in mathematics classrooms, and to try to find ways of limiting these needs. ATD makes DIDIREM team especially sensitive to the distance with the French curriculum and reflect on a possible ecological habitat and niche for Cruislet in the French education system. Another distance with Cassyopée experiment resides in the fact that DIDIREM researchers for whom epistemological concern is a top concern cannot rely for supporting their design on similar stabilized didactic knowledge. The literature regarding the mathematical objects implemented in Cruislet: vectors, systems of coordinates in 3D space, is not developed as that concerning algebra and functions. In such conditions, the kind of controlled design they are used to carry out, trying to create optimal conditions for the interaction between students and some a-didactic milieu and making the mathematical knowledge aimed at emerge from this interaction as an adequate solution to the problem posed, becomes more difficult. The implementation of the first scenario shows the consequences of this loss of control, and how teachers unable to anticipate students' reactions strongly intervene for trying to maintain the designed trajectory.

The solutions found by DIDIREM team and their implementation show another interesting phenomena. The ecological stance leads to emphasize Cruislet potential for connecting school mathematics with out-of-school activities and making students sensitive to the mathematics involved in social technology. The resulting tasks are quite different from the function games proposed by ETL, and formulated in out-of-school terms. A typical one is planning a trip under specific constraints, these constraints being chosen in order to foster mathematical work on vectors, angles and trigonometry. It is worth noticing that these tasks, initially very openly described in the researchers' scenario have been transformed by the teachers with an evident desire of reducing the curricular distance. This is not enough for making anticipations realistic as pointed out above, and students and teachers face the difficulty of copying with such out-of-school tasks completely unusual in the French system. From this perspective, the second experimentation contrasts with the first one. First the conditions, a university workshop, are much less constrained; second, it is visible that the DIDIREM researchers have learnt a lot from the first experiment. Due to these two characteristics, the task design is more appropriate, feedback characteristics which play an essential role in the a-didactic interaction with the milieu are better understood, the balance between what is accessible to students in some adidactic functioning and what has to be conveyed by the teacher is correctly evaluated. Theoretical tools for controlled design are thus again operational.

Another interesting point in the cross-case analysis concerns Logo programming. There is no doubt that Logo programming plays an essential role in ETL design, in full coherence with the notion of half-baked microworld and the constructionist theoretical perspective (Kynigos, 2007b). The tasks proposed by ETL deeply rely on Logo programming, and for instance the task “guess my flight” which plays a central role seems to obey a general game pattern familiar to the team. Conversely, Logo programming is not embedded in the DIDIREM culture, and the tasks they design are inspired by examples provided by ETL colleagues during ReMath meetings. Nevertheless, they adapt these to their particular sensitivities and concerns as explained in the cross-case analysis. This is done in two different ways: in the first experimentation, accent is put on iterative structures, through the programming of flights involving the repetition of the same action, in the second the accent is put on the differential vision of curves related to their curvature properties that Logo makes accessible with a limited mathematical background. In the two cases, we can see there a manifestation of the DIDIREM sensitivity to epistemological concerns.

For concluding this synthetic presentation of the affordances we see in this cross-case analysis for the theoretical reflection, we would like to stress the influence of the theoretical frames on the vision of design that it evidences, revealing how conceptions of design supported by constructionist perspectives on the one hand, TDS on the other hand differ in what they focus on in design work and what they try to control. These contrasting positions also help better understand importance for constructionist perspectives of systems of representations which can support the progressive development of new objects and new representations for these, which is encapsulated in the construct of half-baked microworld. Nevertheless, one cannot forget that TDS from a learning perspective situates within a socio-constructivist perspective. Knowledge is seen as resulting mainly from the adaptation to an adidactic milieu which evolves as far as the student interacts with him. This importance attached to what can be autonomously produced by students through their interaction with the artefacts seen as component of the milieu introduces a possibility of connection between the two perspectives that could be investigated in the future by trying to establish links between central constructs of each of these: adidactic milieu on the one hand and half-backed microworlds on the other hand.

VI. ReMath affordances in terms of theoretical integration

In this part of the deliverable, building on the entire work developed in ReMath and especially on the outcomes presented above of epistemological profiles and cross-case analysis that helped us develop a more synthetic vision of this work, we present what we see as the ReMath affordances in terms of theoretical integration. This presentation uses two levels of description. At the first level, we make explicit what we can consider as a shared theoretical frame among ReMath partners. This STF has been progressively elaborated from the germ presented in the deliverable 13. Voluntarily, for addressing a larger audience, we present it avoiding technical terms as much as possible while at the same time including references to specialized literature. At the second level, we make explicit the connections that ReMath has allowed us to build between theoretical frames and also identify that specific constructs we think useful for approaching these connections. We also make reference to illustrative examples provided by the cross-case analysis.

VI.1. The Shared Theoretical Frame

The ambition of the STF is to express in rather simple terms the positions that the ReMath teams share regarding representations and their role in the learning of mathematics, having especially in mind the learning of mathematics with DDA. As explained in II.3, the first version of the STF was a minimal one. It essentially acknowledged that mathematical objects are only accessible through representations, and that representations have to be thought about as three term objects.

It has been progressively extended along the development of the ReMath project. Drawing on the experience gained, we consider that for being really useful, such STF not only has to be expressed in simple terms as much as possible, but also that it has to focus on what appears to be, in the light of the ReMath experience, a coherent and limited set of positions that we feel important to make explicit. Thinking about what had to be made explicit, we have given priority to positions that fulfil two conditions: we do not perceive these as common sense positions and they deeply influence our vision of the potential of digital artefacts for mathematical learning, and play a major role in our identification of their didactical functionalities.

We thus present below the set of shared positions constituting the STF, then comment and explicit the relationships between these.

Position 1: Mathematical objects are not objects directly accessible to our senses. For accessing them, making sense of them, working on them and exchange with others about them, we use external representations of these objects. We also build internal representations that allow us to access them mentally.

Position 2: It is important to consider representations of mathematical objects in a triadic perspective. A representation is something which stands for something else from someone's point of view. It is also important to acknowledge the social and cultural dimension of representations.

Position 3: The notion of semiotic register of representation is a useful notion for approaching the diversity of systems which have been developed for representing mathematical objects, the rules governing the formation and transformation of representations within such systems, and the crucial role played by connections between semiotic registers in mathematical learning.

Position 4: Meaning-making of mathematical objects and processes is a complex semiotic activity mobilizing a diversity of semiotic systems and semiotic activities. An analysis in terms of semiotic registers of representations pays justice to this diversity only partially. The idea of *multimodality* is a useful idea for approaching the complexity of semiotic systems and activities, thus for understanding the semiotic potential of digital artefacts.

Position 5: Representations of mathematical objects have an essential cultural dimension. Meaning-making of mathematical objects and processes through these representations is essentially a social activity. The idea of *semiotic chain* are a useful idea for approaching the processes of evolution of meanings within the classroom discourse, and understanding the role of mediation tools potentially played by DDAs.

Position 6: For understanding the teaching and learning potential of a DDA or in other words identify its *didactical functionalities*, it is important to identify its potential in terms of representation i.e. analyse DDA's features in terms of representations. This includes its affordances in terms of diversity of representations and ways of dealing with these, in terms of connections between representations, in terms of evolution of representations, and in terms of potential offered to social interactions.

Position 7: When considering the potential of a DDA in terms of representation, it is important to consider that a distinction must be made between didactical functionalities as thought about by the designers of the DDA and those resulting from the interpretations, elaborations and even transformations made by DDA users.

Position 8: When considering the didactical functionalities of a DDA in a specific educational context, it is important to evaluate the distance in terms of representations between the DDA and the educational context at stake.

Position 9: General perspectives on representations can only partially support the identification of didactical functionalities for a given DDA and mathematical domain. Specific epistemological and didactical perspectives and knowledge concerning the domain at stake are also required.

Position 1 expresses the fundamental position according to which mathematical objects are abstract objects, "idealités" and are thus not directly accessible to our senses. This position shared by the different teams does not necessarily express through specific theoretical constructs. This is nevertheless the case for instance in ATD that introduces a distinction between ostensive and non-ostensive objects. It is claimed that mathematical objects belong to the non-ostensive category and thus are only accessible and worked through ostensive objects which can be of very diverse nature (symbolic expressions, graphs, discourse, gestures…). Relationships between ostensive and non-ostensive are considered of a dialectic nature: non-ostensive pilot the use of ostensive but also emerge from these and are continuously shaped by them. Coherently with these views, research in mathematics education has proved the importance for students' conceptualizations to access mathematical objects through a diversity of representations, and the negative consequences on conceptualization of giving too much predominance to one particular system of representation. ReMath DDAs offer for the mathematical objects they implement a rich diversity of representations, both conventional and innovative ones. They thus offer a priori an evident potential for supporting the dialectic development of mathematical concepts and representations.

Position 2 aims at overcoming the limits of the classic dyadic definition of representations relating signifying and signified, through the inclusion of a third pole, by the expression “from someone’s point of view”. This position also expresses the common concern of ReMath teams with context, either internal to the class or external to it. Actually, inside the classroom, it seems fundamental to take into account the possibility of different and potentially divergent points of view, considering each student with respect to the others and with respect to the teacher. Similarly, outside the classroom it seems important to take into account different points of view both with respect to school institutions and the community of mathematicians. The relationship between representation and object is thus dependent on the perspective of the interpreter. This triadic perspective can be expressed in different ways. Several ReMath teams for instance rely on the Piercean vision of semiotics increasingly influential in the field of mathematics education but, while being coherent with such a triadic vision, ATD for instance does not rely on Pierce’s semiotics. Alnuset cross-case analysis is especially insightful for understanding what Pierce’s semiotics offers for understanding the potential for learning of DDAs such as those developed in ReMath.

Position 3 links the notion of representation to that of semiotic register of representation introduced in mathematics education by Duval and used by several ReMath teams. What Duval stresses is that representations of mathematical objects do not only serve their designation. They are functional in the sense that they allow us to work on these mathematical objects. This leads to pay particular attention to the operations which can be carried out on representations, thus to the characteristics of the semiotics systems within which they are produced in that respect. Hence the notion of semiotic register of representation defined as a system allowing three different types of operations: the formation of representation, their treatment within a given register, their conversion in another register, the last one being considered to play a major role in conceptualization as it is essential for discriminating objects from their representations. As pointed out above, ReMath DDAs offer for given mathematical objects a rich diversity of representations. Most of these can be associated to semiotic registers of representations. Moreover, in the design or extension of ReMath DDAs, connection between different systems of representations has been a major focus of attention, thus conversion between representations, which is known to be a requiring task from a cognitive point of view, is supported in different ways. In some cases, once decided, conversions are automatically performed by the DDA, and students can analyze their effects, in other cases, conversions are not automatized but helped by different signs, or their availability can be decided by the teacher. This offers a rich panel of possibilities that the experimentation has not fully explored, and certainly deserves more systematic research.

Position 4 expresses the shared conviction among ReMath teams that due to the diversity of semiotic systems and semiotic activities involved in mathematical activity with DDAs, an analysis in terms of semiotic registers of representations pays justice to this diversity and thus to the learning potential of DDAs, only partially. The idea of multimodality essential in the semiotic vision of some of the ReMath teams as shown by the cross-case analysis seems especially helpful from this perspective. It makes clear that working with mathematical objects people use a wide diversity of cognitive, physical and perceptual resources including oral speech, gestures and various kinds of bodily motion. Understanding how students make sense of mathematical objects and processes through their activity with DDAs requires the consideration of this diversity that goes beyond the diversity of semiotic registers of representations implemented in the DDAs. Note never the less that work on multimodality is still in an emergent state in mathematics education and has been mainly fostered by research concerning mathematical learning in interactive learning environments such as those attached to ReMath DDAs (Edward, Radford & Arzarello, 2009). As pointed out by Sfard (2009), more solid conceptual foundation must be developed for it, deeper understanding about the positive and possibly negative roles of gesturing while learning mathematics has to be developed together with a deeper understanding of the respective affordances of utterances and gestures and of their relationships.

Position 5 expresses the shared assumption that Mathematics is a cultural product. An important part of the activity of mathematicians has been and is devoted to the production of efficient systems of representations allowing them to operate on mathematical objects, and also to systematize and automatize these operations. Representations of mathematical objects are thus a fundamental part of both the process and the product that led to Mathematics. This position expresses the need of regarding the learner not as an isolated element but as part of a larger system to which he/she is going to participate.
A shared assumption concerns also the fact that meaning-making of mathematical objects and processes through representations is essentially a social activity. Thus we recognise the need of broadening the unit of analysis, both in the design and the implementation of any teaching experiment, to include the discourse generated by students and teacher when dealing with tasks related to the use of DDAs. Taking a semiotic perspective, some of the ReMath teams assume specific theoretical approaches that see the central role of representations, either as a product or as a medium, in the teaching-learning process. In particular, when teaching and learning processes are interpreted as the development of classroom discourse in which pupils and teacher are both actively engaged. From this perspective each DDA offers a potential based on its features and different modes of use. The notion of semiotic chain is one of the theoretical constructs that have been fruitfully used in the cross-experimentation and seems to us an efficient tool for describing meaning-making processes in classroom activitities (see for instance the analyses carried out by the UNISI team)

Related to position 5, position 6 concerns the need of identifying the didactical functionalities of a DDA in terms of representations. That means to focus on DDA’s features in order to identify its specific didactic potential related to the different forms of representations available in the software. This includes the analysis of affordances in terms of diversity of representations and ways of dealing with these, but also in terms of connections between representations either internal or external to the DDA. As we saw in the comparative analysis discussed above, ReMath teams shared the interest in the connections between representations. Some of the teams, for instance MeTAH, ITD, and partially UNISI, explicitly exploited the theoretical assumptions (Duval) concerning the use of different semiotic registers to make meaning emerge (Aplusix, Alnuset). The specificity of the semiotic registers available in the different DDAs and the different relationship between these registers and the standard ones, were considered. Other teams focused on the potential offered by the possibility that certain kind of representation have to evolve within a specific DDA. The notion of half-baked microworld, at the core of the design of some ReMath DDAs, expresses such possibility and the didactic potential related to it. The articulation of the different didactical functionalities as they emerge from focusing on the different representations available and according the different theoretical frameworks

According to position 5 and taking into account the social dimension, it is assumed that learners need to make sense of their experience, specifically of manipulating representations, in the context of social interaction with peers and with teachers. For this reason, a specific analysis is needed in order to identify the didactic potential offered to social interactions by the different representations available in a DDA and the ways they can be acted on. Among the ReMath DDAs, MoPIX has a specific place in that respect due to its evolution towards a web-2 based structure, which gives a predominant role to collaborative design and use. This social vision is certainly less central and less controlled in the design of other ReMath DDA, but they nevertheless all offer potential for supporting the social dimension of meaning-making with representations as shown by the design and implementation of the teaching experiments. There is no doubt that, according to the different sensibilities and the different theoretical frameworks, the analysis of this social dimension has been more or less developed. Nevertheless this shared assumption allowed to identify interesting complementarities in the vision of didactical functionalities, as shown above for instance in the cross case analysis related to Casyopée.

Position 7: This position, coherent with position 2, recognises that a particular representation may have different meanings for the designer of a tool and for a particular user. The meaning assigned to a particular element becomes a matter of interpretation and consequently of particular semiotic acts performed by an individual. In particular, while a tool designer may have a clear idea of the ways in which a chosen representation – either a specific element of a DDA or a specific functioning of such element - relates to a specific mathematical object, this does not guarantee that a particular user will see it as such, or even see it as being mathematical at all. This can be linked to the distinction that Rabardel operates between what he calls “fonctions constituantes” and “fonctions constituées” in relation to the process of instrumentalisation of an artefact. It seems worthwhile to point out that the idea of half-baked microworld, a core idea for some ReMath partners, can be seen as a way of offering large possibilities to the user for developing his own “fonctions constituées”. From this perspective, position 7 can be linked to position 6. The two positions, in different but complementary ways, express the importance to be given in the identification of the learning potential of an artefact and the identification of didactical functionalities for it, to the possibilities it offers to the evolution of representations.

Position 8 stresses the necessity of considering the distance in terms of representations between the DDA and the educational context at stake, when considering the didactical functionalities of a DDA in a specific educational context. The importance of this assumption is clearly shown by the experimentation. ReMath DDA have been from the second year of the project classified along an axis regarding this distance, Aplusix and Cruislet occupying the extreme positions. It has also been clear from the beginning of the project that their theoretical frameworks made teams more or less sensitive to this distance, and generated contrasted positions. On the one side, MeTAH and DIDIREM teams relying on ATD were especially sensitive to institutional constraints playing at different levels of determination and this sensibility supported by theoretical constructs has impacted both the design of Aplusix and

Casyopée and the design of the teaching experiments prepared by these teams. There was an evident effort made for maintaining a reasonable and acceptable distance. On the other side, MoPIX, MaLT and Cruislet designers did not seem to pay the same attention to this distance, and their constructionist perspective, on the contrary, led them to emphasize the innovative dimension of their DDA and to foster their use for challenging ordinary practices. In fact, the experimentation clearly showed that whatever the DDA was, its innovative dimension was rather high. It also showed that the idea of distance could only make sense when paired with contextual characteristics, thus the formulation of position 8. The experimentation of the tree representation offered by Aplusix was not easily organized by MeTAH team, even if it could be seen a priori attractive and not so distant from representations actually used in paper and pencil environment, while the sophisticated algebraic line of Alnuset was very well welcome and a long experimentation could be organized by ITD. IoE had to face strong institutional constraints when experimenting MoPIX and MaLT, and the distance of these DDA with standard tools and practices imposed uses in specific and constrained conditions, while the same distance did not generate the same difficulties for ETL.

Position 9 stresses that general perspectives on representations can only partially support the identification of didactical functionalities for a given DDA and mathematical domain. It attracts our attention on the fact that specific epistemological and didactical perspectives more directly related to the domain at stake are also required. The domains covered by ReMath DDA were diverse from algebra and functions to 3D geometry, and mechanics. A huge amount of didactical and epistemological research has accumulated regarding the teaching and learning algebra and function, and rather consensual positions can be expressed today in these domains. The way this knowledge has impacted both the design of Aplusix, Casyopée and Alnuset, and the associated teaching experiments is evident. It makes the identification of didactical functionalities more precise and argumented in a more convincing way. It makes possible learning outcomes of the experiments easier to anticipate and identify, what is certainly of importance for the wide dissemination of such DDAs. On the side of 3D geometry and mechanics, research exists of course but not to the same extent, and a clear vision of what can be achieved and why is not so evident. These differences, beyond those resulting from differences in theoretical approaches, affect the design of teaching experiments, the expectations put on these and their a posteriori analysis.

Coming to more general comments on our shared definition of representations and the visions it has supported or might support, we would like to stress the following points:

The generality with which in the definition of representations we have expressed the relationship between signifier and signified - “something which stands for something else” – intends to convey the shared idea that the semiotic relationship should be considered in a very broad sense. Thus not only generic material elements such as inscriptions[4], but any other element, no matter the expressive means that is used. A quite usual kind of elements, that cannot be reduced to inscriptions are oral utterance and gestures, but we purposefully intended to be as large as possible because of the novelty of the phenomena that we intended to investigate.

[4] ‘Inscriptions are signs that are materially embodied in some medium, such as paper or computer screens.’ (Roth & McGinn, 1998, p. 37).

On the one hand, accordingly with the basic assumptions of the ReMath project we recognized and intended to exploit the relationship between DDAs (or their specific parts) and mathematical objects. On the other hand, we intended to foster and exploit semiotic processes related to the use of the new media, different from those that are the typical of the mathematics practice.

Never the less we are aware of the peculiarities of certain types of representation, such as classic mathematical notations. A mathematical object exists as pointed out above as a cultural product, whose properties are agreed within some community of mathematicians (however that may be defined). It is produced and reproduced through the use of inscriptions of various kinds in a range of modes through different media, including paper-and-pencil and computational media. These are considered to be representations of the object when they become shared for members of the community, incorporated into the system of meanings associated with the object. Such representations form networks of related elements with rules of combination and transformation, including conversion between different systems of representation (e.g. algebraic notation and Cartesian graphs).

As clearly pointed out by Kaput, also referring to (Goldin & Kaput, 1992, 1996):

> "external representations" occur in *systems*. Such systems, while they can be personal and idiosyncratic under certain circumstances, usually are cultural artifacts that cannot be separated from what is normally taken as "mathematical content." Indeed, learning such systems and how to operate within them dominates school mathematics. Moreover, such systems are seldom used singly or in isolation from one another. Most mathematical activity involves multiple representation systems used in combination with one another." (Kaput, 1996)

Therefore, in spite of the generality of the shared definition, we recognize the need of keeping in mind the specificity of particular kind of representations in mathematics. Specificity may concern the medium, for instance paper and pencil with respect to a computer screen, or the familiarity in the use, for instance standard linear notation in respect to tree representation for algebraic expressions. That shared concern brought our teams to take explicitly into account in their research questions the relationship between new forms of representation based on the use of the DDA and classic forms, primarily standard notation in paper and pencil.

The generality of our formulation allowed us to consider the idea of representation –both as process and product – in its generality, and let each team the possibility of developing differently the three pole of its structure according to the specific concerns inspiring its own investigation.

For instance, some of the teams were interested in studying the semiotic processes related to the role of tools, specifically ITC tools, in learning and teaching. Hence, in these cases, the minimal theoretical frame consisting in the shared definition of representation was further elaborated by the team through the definition of more specific notions. For instance, the UNISI team introduced the Theory of semiotic mediation (Bartoini Bussi & Mariotti, 2008) both in the design of the teaching sequence and in the analysis of the collected data. According to that theory the notion sign was utilized in a sense deeply inspired by Pierce, and consistent with the latest claims concerning the need of enlarging the notion of semiotic system (Radford 2003; Arzarello 2006) including different and more flexible kind of signs.

Elaborating on this notion of sign the theoretical framework was developed by defining the notion of semiotic chain as tool of analysis to describe the teaching and learning process as the evolution from personal signs towards mathematical signs. In social interaction process signs emerge through discourse, leading teachers and learners to share mathematical meanings.

Other teams, for instance MeTAH team, developed the common MTF through the use of Duval's notion of semiotic register: in particular, in the design of the tasks, they elaborated on his epistemological stance concerning the role of specific semiotic processes – conversion and treatment – in the construction of knowledge (Duval, 1995).

The idea of semiotic system has central importance also within systemic functional linguistics, which sees a semiotic system as a set of components (lexicon), combined using a particular set of rules (grammar). Different semiotic systems have different potential for meaning making (O'Halloran, 2005; Kress, 2001), more precisely meaning is constructed through the choices made by the individual between elements of a semiotic system. Hence the articulation of the idea of representation as "something which stands for something else from someone's point of view" still fits well the systemic functional linguistics perspective.

Within this perspective, the availability of different semiotic systems is also valued. In fact, multi-semiotic environments allow participants many opportunities for making meanings with the representations available to them and choices about the most apt representations to employ in order to communicate their desired meanings. In a multi-semiotic environment, meaning potential is broadened by the conjunction of the meaning potentials of the various systems; meanings are constructed through choices both within and between systems.

In the meaning making process, specific attention is paid to the possible role of everyday discourse. For instance, when the pedagogical plan entails activities based on (or evoking) a 'real world' context, the classroom discourse will be enriched with elements of the discourse from outside, which have meaning for students. They will thus be likely to draw on everyday discourses and forms of representation as well as on the formal mathematical discourses and representations encountered in the classroom. This provides opportunities for them to form links between these discourses, enabling sense to be made of the representations that are new to them. Such links between different domains of activity may occur through the formation of chains of signification, where similar signifiers are encountered in different discourses (Carreira, et al, 2002).

Summarizing, the efforts made towards the definition of a Shared Theoretical Frame, expressed simply and oriented towards the practice of research and design, made it possible to overcome fragmentation, have a clear view of our similarities and differences. While preserving our respective coherences and identities, it also showed us the way we could benefit from visions different from ours in a practical way, leading to the idea of Connected Theoretical Landscape that we present in the next part of this deliverable.

VI.2. The Connected Theoretical Landscape

When the ReMath STREP started, the teams involved in it were aware of the diversity of theoretical perspectives represented in it. They were also aware that most of the teams did not rely on only one frame, what means that they had already undertaken some kind of networking between theories up to the stage of combination and partial integration. But these constructions had been realized within given groups sharing similar backgrounds and facing similar educational contexts. A already pointed out in part III, the challenge of ReMath was of a different nature, having the ambition of extending networking towards teams with different educational background and submitted to different contexts.

Substantial advances have been reached in that direction. They have made possible the syntheses presented in parts IV and V and the elaboration of the Shared Theoretical Frame described above. In this last part, we present another and complementary attempt at expressing these advances, in terms of Connected Theoretical Landscape. Let us consider the initial state of the theoretical ReMath landscape as shown in figure 1.

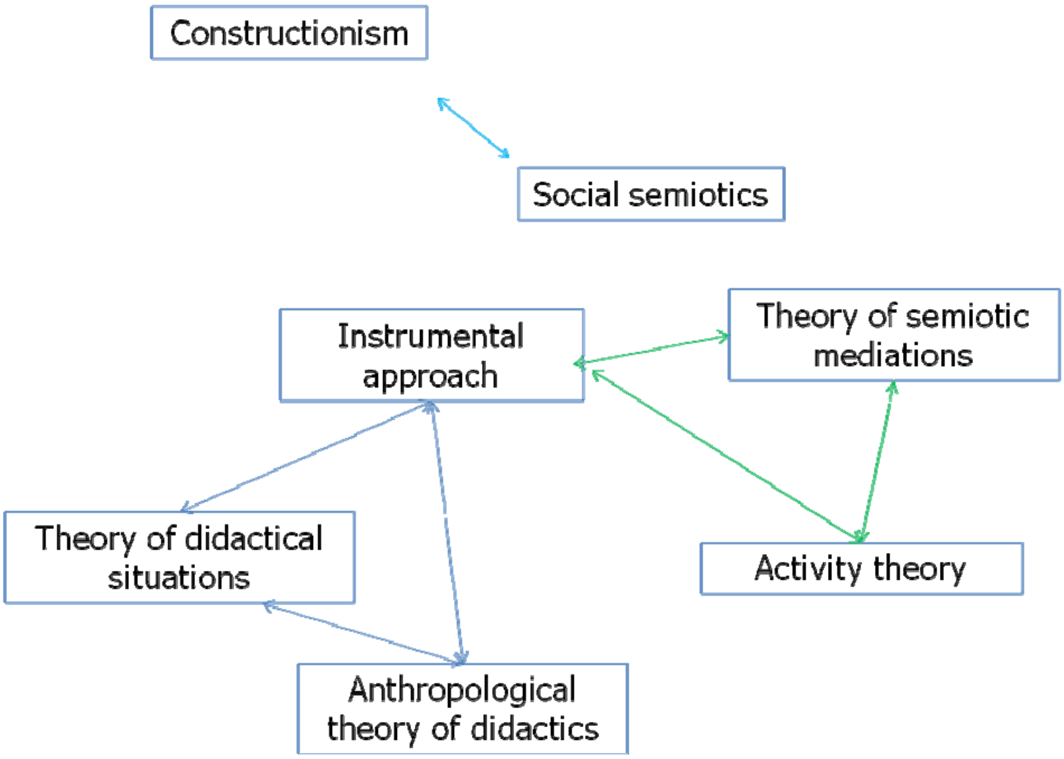

Figure 2: The initial theoretical landscape

Different theoretical frames are present in it and some well established connections are already there. These are essentially connections active at the level of a particular team or educational culture. For instance, the blue connection between Constructionism and Social Semiotics is a connection active in IoE team; the red connection between the Theory of Didactical Situations, the Anthropological Theory of Didactics and the Instrumental Approach is active in DIDIREM and MeTAH teams; the green connection between Activity Theory, the Theory of Semiotic Mediation and Instrumental Approach is active in UNISI team. These associations between teams, theoretical frames and connections are visualized in figure 2.

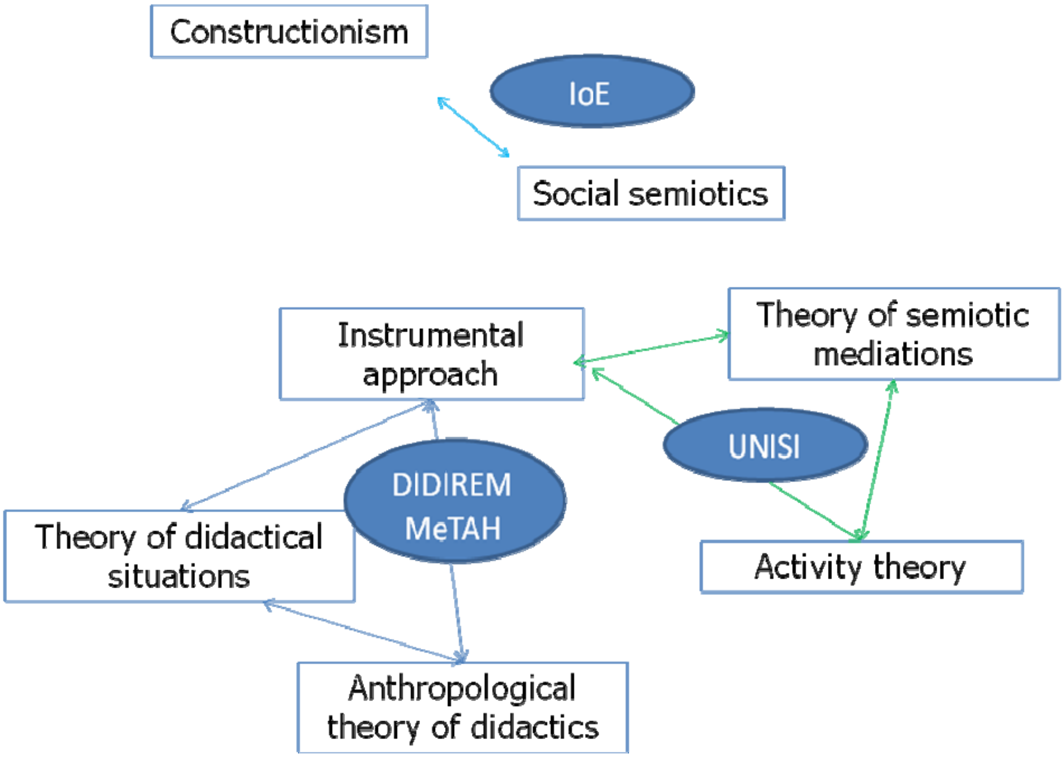

Figure 3: Connections and Teams

The work carried out in ReMath has enriched this landscape in different ways. Thanks to the cross-experimentation, thanks to the reflective work carried out through the elaboration of DDA epistemological profiles and cross-case analyses, the different teams have better understood the interest of constructs outside their culture and the way they could make sense and benefit of these while respecting their specific coherence. Some particular constructs have emerged with specific potential for supporting complementarities and connections. The specific coherence inherent to each approach has also been better perceived. In the following, we synthesize this evolution, focusing on some emblematic points. More details and carefully selected illustrative examples can be found in the cross-case analyses presented in Appendix 2.

Complementarities and connections between the theory of didactical situations and the theory of semiotic mediations

These have been well evidenced by the Casyopée cross-case analysis. In the synthesis presented in part V, we have focused on the interest of complementing the constructs provided by TDS by those coming from TSM for understanding the possible progression of knowledge in collective discussions following adidactic phases of students' autonomous work and the way the teacher can help this progression through processes of semiotic mediation. The comparison between two collective phases coming from the two experimentations in particular makes clear the difference between collective discussions with the meaning given to these in TSM, and institutionalization with the meaning given to this term in TDS. In collective discussions, the teacher is asked to foster students to elaborate on their own personal meanings: the evolution of students' personal meanings has to be based on previous activities with the artefact and personal meanings having emerged. No classroom discussion in the TSM sense would be possible without the active engagement of the students, for instance for making explicit the meanings they personally developed, sharing them with classmates and the teacher. This semiotic process that is going to occur during a collective mathematical discussion involving the teacher and the students, the artefact and the mathematics, is the core of the process of Semiotic Mediation assumed by the TSM. The crucial point is how to relate shared meaning with mathematical meanings that for their nature are not negotiable but culturally established. In this respect the role of the teacher becomes fundamental as well as delicate, and different of her role as modelled when collective phases are considered as moments of intitutionnalization. In these, the teacher is asked to manage the tuning of knowledge produced by students in the adidactic situation with that aimed at by the institution and organize its decontextualization. The semiotic processes potentially engaged in this tuning and decontextualization are not paid the same attention as in TSM and are not treated with the same level of theoretical control. In fact, as pointed out in the cross-case analysis, in TDS the notion of situation of formulation and the associated dialectics are asked to take this semiotic evolution in charge from a theoretical point of view. In practice, very often the conditions for the development of this dialectics are not set up and what is observed is a succession of adidactic situations of action and collective phases labelled as institutionalization phases but cumulating formulation, validation and institutionalisation aims. As explained in the cross-case analysis, time constraints contribute to this phenomenon. There is no doubt that TSM thus helps us understand the limitations of engineering designs relying on TDS when this dialectics of formulation is not established properly and the resulting effects. Conversely, TDS constructs and more especially the notion of didactic contract and the phenomena associated to its paradoxes such as Topaze effects are helpful for analyzing collective discussions. As shown by the episodes analyzed in the cross-case analysis, in collective discussions both students and teacher contribute to the progression of knowledge. This requires and adequate sharing of responsibilities between students and teacher, thus a specific didactic contract, and a lot of expertise from part of the teacher for piloting processes of semiotic mediation without falling into the tramps of Topaze effects. TDS makes us sensitive to the necessity of establishing such a contract and helps to identify its characteristics. Moreover it provides conceptual tools for questioning the genuine character of observed collective discussions.

The identification of these complementarities directly results from the material used for this cross case analysis focused on collective phases. Let us add that the Casyopée cross-experimentation has also shown the interest of complementing the constructs of the theory of semiotic mediations by constructs coming from TDS. For instance, it has shown the interest of the conceptual tools provided by TDS for anticipating the adidactic potential of the tasks proposed to the students thanks to a detailed a priori analysis of the possible interactions of the students with the adidactic milieu and of the feedbacks resulting from this interaction. The Aplusix cross-case analysis pays specific attention to this notion of feedback, comparing the use made of the feedbacks provided by Aplusix by MeTAH, ITD and UNISI teams respectively shapes by TDS+ATD, Activity Theory and TSM perspectives. More globally, these theoretical constructs have proved their efficiency in the design of tasks appropriate to well identified educational goals, in the cross-experimentations involving the DIDIREM and MeTAH teams, which are well familiar with these theoretical constructs and their operational use.

Combining and enriching instrumental perspectives

All ReMath teams more or less refer to Rabardel's ideas which are at the source of the Instrumental Approach. Thus the distinction between artefacts and instruments, the notion of instrumental genesis and the duality of the processes of instrumentalization and instrumentation it involves are shared ideas among ReMath teams. These shared references coexist with an evident diversity regarding the perception of instrumental issues. The fact for instance that the instrumental approach developed with the DIDIREM team is a combination of Rabardel's ideas and ATD gives the vision of instrumental issues developed by this team a coloration different from that developed by ETL whose basic theoretical perspective is constructionism, and from that developed by UNISI whose basic theory is the heory of semiotic mediations. These differences have been made clear by the cross-case analysis, and contribute to enrich our vision.

For instance, Casyopée cross-case analysis has shown the impact of these differences in the way decontextualization was conceived. For DIDIREM, what students develop through their work on mathematical tasks with Casyopée are praxeologies in which Casyopée is deeply incorporated. Associated techniques are instrumented techniques and the technological discourse is expected to intertwine mathematical and artefact knowledge. Decontextualization obeys the same characteristics. Even if Casyopée instrumented techniques are carefully connected with paper and pencil techniques, both of these are taken into account in the decontextualization process in a similar way. For UNISI, Casyopée is perceived as a tool for semiotic mediation, and thus decontextualization is conceived as a double process where students have to situate the task already solved into a larger class on the one hand, detach the solving process from Casyopée use on the other hand. For that reason, decontextualization takes place without access to Casyopée.

ETL team offers also an interesting and original perspective on instrumental issue due to the fundamental role it gives to the notion of half-baked microworld. This fundamental role is explicitly related by ETL to Rabardel's approach. More precisely, half-back microworld are expected to foster creative instrumentalization processes from part of the students, and thus contribute to productive instrumental genesis. As pointed out in MoPIX cross case analysis, half-baked microworlds are designed for instrumentalisation since they invite the students to deconstruct them, change them and build on their parts, engaging in the way in exploration and construction activities, rich in the generation of meanings. This way of operationalizing the creative dimension of instrumentalization is original when compared with the ordinary uses of the instrumental approach and certainly enrich our vision of it. There is no doubt that it is highly influenced by the constructionist perspective adopted by ETL.

Connecting different semiotic perspectives

As evidenced by the high level commonly attached to the semiotic concern, all ReMath teams are sensitive to semiotic issues but it is also clear that they approach these issues from a diversity of theoretical perspectives, using a diversity of theoretical constructs. Once more, ReMath work has helped to establish productive connections and complementarities between these different perspectives, leading to shared visions which have been synthesized in the Shared Theoretical Frame. We would like to insist more here on the way can be productively exploited. ReMath teams agree for instance on the interest of the notion of semiotic register of representation due to Duval even if they give to it a more or less important role in their designs and analyses. This notion is adequate for describing most of the representations of mathematical objects implemented in ReMath because these satisfy the criteria associated to semiotic registers: being structured systems allowing the formation of representation according to precise syntactic rules, their transformation within the same register and the conversion towards other registers. This notion of semiotic register was productively used by several teams for DDA design, for the identification of didactical functionalities, and the building of pedagogical plans. But, what also made clear by the cross-experimentation, is that this notion needs to be complemented if one wants to understand the diversity of semiotic processes contributing to meaning making with DDAs such as those developed in ReMath. From this perspective, Alnuset cross-case analysis shows with a lot of insightful details how the Piercean semiotics can be fruitfully combined with Duval's approach. MoPIX and MaLT cross-case analysis are also especially insightful from this point of view, showing the pertinence of a vision in terms of multimodality. This idea of multimodality initially especially supported within ReMath teams by social semiotics has thus progressively become a shared object contributing to the connection of our different perspectives regarding representations.

Connections between constructionism and social semiotics

Connections between constructionism were already present in the initial theoretical landscape, internal to IoE team members, but the cross-experimentations involving both ETL and IoE led to a deepening of the reflection on these connections. This was achieved through two cross-case studies concerning respectively MoPIX and MaLT, and connections established more especially concern the similarities and differences between the two approaches regarding meaning making. These connections have been facilitated by the fact that constructionism is not perceived by ETL as an individualistic theory that views the meaning generation process as being detached from the students' interactions with their social environment and as a synonym to the cognitive development, achieved merely through the individual's interactions with the given computational tools. This is made clear by the importance given to the notion of half-baked microworld and the way this is described and used by ETL teams. Beyond the support they offer to instrumentalization recalled above, half-baked microworls are also perceived as boundary objects, in the sense that they can convey meaning among of the members of the same community, operating as a tool of communication, around which the members of the community organise their activities. The generation of meanings is considered to emerge and shaped both by the student's mathematical activity as they interact with the available tool (the microworld and its representations) and their social activity supported the half-baked microworld's own characteristics. ETL takes thus a socio-cultural view of the use of the representations afforded by computational tools as means to transform and re-organise mental processes taking place in specific social settings, which makes the connection with social semiotics possible and productive. A critical point in this process concerns the construction of new meanings and the creation of space –by tool and task design- for the students to move the focus of their attention to new objects and relations emerging from their interaction with the available tools. MoPIX and MaLT cross-case analyses show how the design of tasks and their management in the classroom is organized in ordre to make these constructions and moves likely to appear.

The IOE team is similarly interested in studying the ways students interact with available representations and the ways in which representations structure meaning making. But, as pointed out in the cross-case analyses involving both ETL and IoE teams, there are two principal ways in which the IOE multimodal social semiotic approach to this interest differs from that of the ETL team.

First, the notion of meaning making itself is conceived as located only in the discursive moment as students (and teachers/researchers) interact with each other and with available resources, including technological tools, other forms of representation and discursive resources drawn from previous experiences. There is no supposition of shaping of mental processes but only analysis of the ways in which the various resources are brought together into coherent texts or analysis of possible sources of lack of coherence. The analysis focuses on the chains of signification giving rise to particular ways of speaking about the objects at stake. As made clear in the cross-case analyses, this conception of meaning does not allow direct statements to be made about the cognition of individuals or about learning. In order to consider learning it is necessary to define this in terms of change in patterns of interaction. This is not the case for socio-constructionism as evidenced by the cognitive interpretations made of students' behaviors by ETL teams.

The second difference between the approaches to meaning making lies in the weight given, both in the design of tasks and in the analysis of data, to the role of the representations offered by the DDAs. While a principal purpose of the cross-experimentation was clearly to evaluate the use of these, the multimodal perspective challenges the primacy of any one form of representation, implying that any analysis must attend to choices and interactions between representational systems. For the IOE team, the use of a range of both traditional and innovative representations is thus a fundamental principle of both the pedagogical plan and the research design.

Identifying better the similarities and differences between these two approaches, and the way these can mutually contribute to our understanding of meaning-making through representations has been an evident result of the cross-experimentation process involving IoE and ETL teams as shown by the cross-experimentations involving the two teams.

Connections between constructionism and the theory of didactical situations

The possibility of establishing productive connections between constructionism and the theory of didactical situations were not evident at all when the ReMath project started. The characteristics of Cruislet, the DDA experimented both by ETL and DIDIREM teams whose experimentation was supposed to foster these connections, the difficulties met by DIDIREM with the organization of its experimentation reinforced first this pessimistic view. Never the less, interactions along the ReMath project and the development of the cross-case analysis helped us reconsider this position.

Reflecting on what has been achieved, and complementing the synthesis of the cross-case analysis in part V, we would like to insist here on some important points. There is no doubt that the characteristics of Cruislet, without any doubt the DDA the most distant from the software usually used in mathematics education, played an important role, destabilizing

DIDIREM approaches and practices. This destabilization and the way it was managed contributed to make clear what was really fundamental in the theoretical approaches underlying DIDIREM work and what could be negotiated. As as been pointed out in the synthesis of the cross-case analysis, the choices piloting the design of the Cruislet experiment were quite different from those guiding the design of the Casyopée experiment. Particular attention was paid in the definition of educational goals to the proximity of Cruislet with out-of-school technology. Priority was thus given to open and realistic tasks from an out-of-school perspective, in the form of challenges and games, what can be seen not so far from ETL views. The innovative dimension of Cruislet and the novelty of the tasks envisaged made anticipation of students' behavior and possible cognitive outcomes more difficult. What is interesting when thinking about connections between constructionism and TDS is that this difficulty did not affect the two teams in the same way. ETL team seemed quite at ease with such a situation, and coherently with the constructionist perspective it relies on, did not fix precise educational goals. What was essential was to ensure that sufficient space would be open to the students for productive interaction with the DDA and meaning making activity through the pedagogical plan envisaged. For the DIDIREM team, the situation was not so comfortable, and this discrepancy contributed to clarify some essential dimensions of design inspired by TDS (usually labeled as didactical engineering (Artigue, 1989, 2009). In didactical engineering, the anticipations made in the phase of analysis a priori play a major role in design. The design tries to control and optimize the characteristics of the interaction between the students and the milieu, including here the DDA, through a careful choice of the didactic variables of the tasks proposed to the students and their management. The design also tries to anticipate what can be an optimal sharing of mathematical responsibility between the students and the teacher, and what didactic decisions can help maintain this optimal situation if difficulties appear. Such anticipations were very difficult for the first experiment carried out, and seriously affected the implementation. It is worth noticing that the experience gained in the first experiment made the context different for the second experiment, leading to a design efficiently inspired by a convincing a priori analysis. Finally, along this experience DIDIREM opened its vision of the definition of educational goals but maintained its strong attachment to the fundamental values underlying engineering design in TDS: the importance given to a priori analysis inspired by epistemological concerns, the vision of design as a controlled process through the anticipation and regulation of the didactic variables of the tasks, the interaction with the adidactic milieu, and the optimized sharing of mathematical responsibilities between students and teacher in the classroom.

There is no doubt that constructionist perspectives do not convey a similar vision of design. Nevertheless, considering the tasks developed by ETL and DIDIREM teams, there is no doubt that in both cases we see the influence of a socio-constructivist vision of learning through the interaction with an appropriate milieu. In each case, we see that the design is expected to ensure that these interactions will be reasonably productive from a cognitive perspective. In both cases it is considered that interactions have to be organized in their social dimension. What really differs is what each form of design ambitions to control. In both cases as stressed above the conditions for productive interactions have to be controlled. But TDS wants this control to extend to hypothetical learning trajectories coherent with the epistemological vision of designers. This is much more requiring in terms of design. When successful, it evidences a high level of theoretical control over didactical phenomena, but as also shown by ReMath cross-experimentation when anticipations are made difficult by different factors it can lead to implementations where the forced realization of anticipated trajectories can become an obstacle to genuine meaning making activity. Hence the importance of design regulation in itinere.

Theoretical frameworks and contextual issues

In the ReMath project, contextual issues, both at a local level and at a more global level have played an important role. The different teams have tried to adapt to the constraints generated by contextual factors, and we have observed that the restrictions they imposed were very different from one context to the other one. The effects of contextual characteristics of the different experimentations have been addressed in part V and we do not come back to the details of the analysis already presented. In this paragraph, we would like to envisage these contextual issues from a theoretical perspective, considering two different points: the influence of theoretical frameworks on the vision of contextual issues on the one hand, the theoretical approach of context on the other hand.

The DDA design and the cross-experimentation have shown that theoretical frameworks influence the way contextual issues are perceived. As has been made clear, decisions (regarding DDA design or the selection of didactical functionalities for these) often result from a complex intertwining of theoretical and contextual factors. From this point of view, the contrast between design choices underlying the development of Aplusix and Casyopée on the one hand, and MaLT and Cruislet on the other hand is evident. In the first case, we observe a development taking place in a highly constrained educational systems while in the second case the constraints seem more flexible, at least as far as innovative practices are concerned. As a consequence, DIDIREM and MeTAH pay a high attention to curricular distance, trying to keep it reasonable, while ETL situates MaLT and Cruislet design and experimentation in a radically innovative position, DDAs and their use being asked to challenge usual practices. But such decisions do not only result from contextual characteristics. MoPIX design also takes place in a quite constrained environment and does not obey the same strategy as Aplusix and Casyopée design. Theoretical frameworks also play a fundamental role. From this point of view, constructionism and TAD certainly are the most distant constructions. And the fact that DIDIREM and MeTAH teams rely on TAD on the one hand, while ETL and IoE base their DDA design on constructionism on the other hand is not at all an accident. We think that in ReMath, thanks to the diversity of theoretical and contextual situations we faced, thanks to the exploitation we have made of this diversity, we have made substantial progress in the understanding of the way theoretical frameworks and contexts intertwine for influencing DDA design, the identification of DDA didactical functionalities, and their implementation in classrooms.

This being said, a question remains open, that of a theoretical approach to contexts, as it seems that in many cases, contextual issues have been taken in charge in ReMath practically, not under theoretical control. Among the theories referred to by ReMath teams, ATD certainly is, the theory having paid the greatest attention to contextual issues, at the global level. This directly results from the fact that its basic object are institutions, that practices are seen as institutional objects, and that meaning making processes even at individual level are considered embedded in institutional practices and shaped by these and the systems of values underlying them. This has certainly contributed to the high sensitivity expressed by DIDIREM and MeTAH teams to contextual characteristics at the global level. Nevertheless, it seems that the whole potential offered by ATD for approaching contextual issues has not been used in ReMath. There is no reference made for instance to the hierarchy of levels of determination (Chevallard, 2002) through which the theory tries to discriminate the different nested influences didactical systems are submitted to. More work has certainly to be done in order to develop an appropriate theoretical approach of contexts, adequately linking the global and local levels we have distinguished in ReMath.

Thus finally, at the end of the ReMath project, the theoretical landscape presented above has substantially changed. Figure 4 tries to express these changes but succeeds very partially in conveying the richness of connections which have been built. For avoiding too many links, the shared objects in blue have not been connected to the theoretical frameworks but have to been as mediating objects connecting the theoretical frames at the periphery of the landscape and complementing the connections between these frames already made visible.

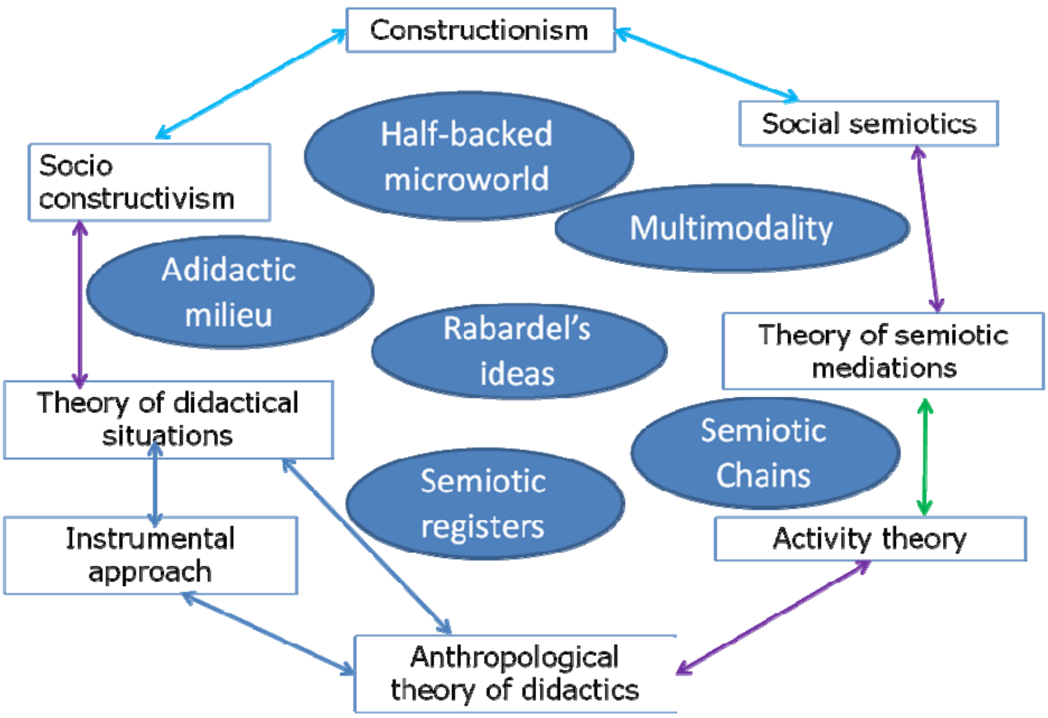

Figure 4: Connected theoretical landscape

VI.3. Conclusion

In the ReMath project, we have addressed the difficult issue of the fragmentation of theoretical frameworks in a specific context, that of learning and teaching mathematics with digital dynamic artefacts, with a specific focus on representations. Our reflection has been supported by the design or extension of six different DDAs, and a sophisticated process of cross-experimentations. Building on the existing literature and on the experience gained within TELMA, we have developed a specific idea of theoretical integration, in terms of connections and complementarities preserving the essence and coherence of the different theoretical perspectives represented in ReMath. For better understanding the needs and possibilities, we have built a first version of the ITF based on the notion of didactical functionality and the meta-language of concerns. This methodological tool has been put to the test in the cross-experimentation process and complemented by several other tools.

At the end of the ReMath project, what we present as the final outcome of it in terms of theoretical integration is rather different from this first version of the ITF as could be expected. It is made of a Shared Theoretical Frame articulated around 9 shared positions, and a Connected Theoretical Landscape trying to make clear the most productive connections and complementarities evidenced by the ReMath project. These are organized around boundary objects which support the communication between the different theoretical perspectives and whose interest is acknowledged by all ReMath teams. These two main outcomes are complemented by two syntheses: the synthesis of DDA epistemological profiles and the synthesis of the cross-case analysis. Through these syntheses, our intention is to help the reader understand what really underlies the STF and the CTL, and how theoretical integration practically can work. We have made particular efforts in order to make our reflection and its outcomes useful to a larger audience, and hope to have at least partially succeeded. Theoretical integration is a very challenging task and making full sense of the outcomes reached in the integration of the diversity of theoretical approaches we have faced in ReMath will certainly be difficult to those who have a superficial understanding of these theoretical approaches. We nevertheless hope that the way we have approached the issue of theoretical integration, focusing on the operational dimension of theories, and through the notion of didactical functionality connecting tightly the analysis of DDA characteristics, the definition of educational goals and the elaboration of modalities of employment in form of pedagogical plans, will help the reader to develop the practical vision of theoretical integration that has inspired us along the years of the ReMath project, and make it productive as this has been the case for all of us.

References

Artigue, M. (1989). Ingénierie didactique. *Recherches en Didactique des Mathématiques*, vol. 9.3., 281-308.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*. Vol 7 (3), 245-274.

Artigue, M (2008) Digital technologies: a window on theoretical issues in mathematics education, in Pitta- Pantazi, D. and Philippou, G. (eds.) *Proceedings of CERME 5*, Larnaca, Chyprus, http://ermeweb.free.fr/CERME5b , 68-82.

Artigue, M. (2009). Didactical design in mathematics education. In, C. Winslow (Ed.), *Nordic Research in Mathematics Education. Proceedings from NORMA08 in Copenhaguen*, pp. 7-16. Rotterdam: Sense Publishers.

Arzarello F., Bazzini L., Chiappini G., (2002), A Model for analysing algebraic processes of thinking, in R. Sutherland et al. (Eds.), *Perspective on school algebra*, Dordrecht: Kluver Academic Publisher, 61-81.

Arzarello, F., Paola, D. Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97-110.

Bartolini Bussi, M.G., and Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In L.English, M. Bartolini Bussi, G. Jones, R. Lesh, and D. Tirosh (eds.) *Handbook of International Research in Mathematics Education*, second revised edition, Lawrence Erlbaum, Mahwah, NJ.

Bottino, R. et al. (2007). *Scenario Design, First Version*. ReMath Deliverable 7.

Brousseau, G. (1997) Theory of didactical situations in mathematics, Dordrecht, Kluwer Academic Publishers

Carreira, S., Evans, J., Lerman, S., & Morgan, C. (2002). Mathematical thinking: Studying the notion of 'transfer'. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 185-192). Norwich: School of Education and Professional Development, University of East Anglia.

Cerulli, M., Pedemonte, B., and Robotti, E. (2006). An integrated perspective to approach technology in mathematics education. In, M. Bosch (ed.), *Proceeding of CERME 4*, pp.1389-1399. Barcelona: FUNDEMI IQS Universitat Ramon Lull.

Cerulli, M., et al. (2008). Comparing theoretical frameworks enacted in experimental research: TELMA experience. *ZDM. Comparing, Combining, Coordinating – Networking Strategies for Connecting Theoretical Approaches.* 40 (2), 201-213.

Chevallard, Y. (1992), Concepts fondamentaux de la didactique. Perspectives apportées par une approche anthropologique. *Recherche en didactique des Mathématiques* 12(1*)*, 73-112.

Chevallard, Y (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*. 19 (2), 73-112.

Chevallard, Y. (2002). Organiser l'étude 3. Écologie & régulation. In: Dorier, J. L. et al. (Eds.), *Actes de la 11e école de didactique des mathématiques*. pp. 41-56. Grenoble: La Pensée Sauvage.

Cobb, P. (2007). Putting Philosophy to Work : Copying With Multiple Theoretical Perspectives. In, F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning,* 3-38. Information Age Publishing, Inc., Greenwich, Connecticut.

Cole, M., Engestrom, Y., 1991, A cultural-historical approach to distributed cognition. in G. Salomon (ed.), *Distributed cognition*, Cambridge: MA, 1-47.

DfES. (2001). *Key Stage 3 National Strategy - Framework for Teaching Mathematics: Years 7, 8 and 9.* London: Department for Education and Skills.

Douady R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches en Didactique des Mathématiques*, 7 (2), 5-32.

Drouhard J.-Ph. (1995), Algèbre, calcul symbolique et didactique. In R. Noirfalise, M.-J. Perrin-Glorian (Dir.), *Actes 8ème École d'Été de Didactique des Mathématiques*, Clermont-Ferrand: IREM.

Duval, R. (1993), Registres de représentation sémiotique et fonctionnement cognitif de la pensée, *Annales de Didactique et de Sciences Cognitives* 5, IREM de Strasbourg.

Duval, R. (1995). *Sémiosis et pensée humaine*. Bern: Peter Lang.

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in the Learning of Mathematics. *Educational Studies in Mathematics, 61*(1 - 2), 103-131.

Eckstein, S. G., & Shemesh, M. (1989). Development of children's ideas on motion: intuition vs. logical thinking. *International Journal of Science Education, 11*(3), 327-336.

Edwards, L. , Radford, L., Arzarello, F. (Eds.) (2009). Gestures and Lutimodality in the Construction of Mathematical Meaning. *Educational Studies in Mathematics*. Special Issue. 70 (2).

Filloy E., Rojano T., Rubio G. (2000). Propositions concerning the resolution of arithmetical-algebra problems. In R. Sutherland et al. (Eds.), *Perspectives on school algebra*, Dordrecht: Kluwer Academic Publisher, pp. 155-176.

Graham, T., & Berry, J. (1990). Sixth form students' intuitive understanding of mechanics concepts. *Teaching Mathematics and its Applications, 9*(2), 82-90.

Harel, G., & Papert, S. (Eds.). (1991). *Constructionism*. Norwood, NJ: Ablex.Herbst, P., Brach. C. (2006). Proving and ‘doing proofs’ in high school geometry classes: What is ‘it’ that is going on for students and how to they make sense of it? *Cognition and Instruction,* 24. 73-122.

Hersant, M., Perrin-Glorian, M.J. (2005). Characterization of an Ordinary Teaching Practice with the Help of the Theory of Didactic Situations. *Educational Studies in Mathematics*, 59 (1-3), 113-151.

Hoch, M. & Dreyfus, T. (2006). *Stucture sense versus manipulation skills: an unexpected result.* Proceedings of PME 30. Prague. Vol 3, pp. 305-312.

Kafai, Y., & Resnick, M. (Eds.). (1996). *Constructionism in Practice: Designing, thinking and learnign ina digital world*. Mahwah, NJ: Lawrence Erlbaum Associates.

Kieran, C. (1989), The Early Learning of Algebra: A Structural Perspective. In S. Wagner and C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra*, Lawrence Erlbaum Associates, pp. 33-56.

Kieran, C. (2004). The Core of Algebra: Reflections on its Main Activities, in Stacey et al. (eds.), *The Future of the Teaching and Learning of Algebra: the 12th ICMI Study*. Springer.

Kress, G., Jewitt, C., Ogborn, J., & Tsatsarelis, C. (2001). *Multimodal Teaching and Learning: The rhetorics of the science classroom*. London: Continuum.

Kress, G., & van Leeuwen, T. (2001). *Multimodal Discourse: The modes and media of contemporary communication*. London: Arnold.

Krotoff A. (2008), *Alnuset, un micromonde d'algèbre dynamique. Analyse didactique dans le cas de l'étude des notions de fonction, d'équation et d'inéquation*. Mémoire de Master 2, Université J. Fourier, Grenoble.

Kynigos, C. (2007a), Half-Baked Logo Microworlds as Boundary Objects in Integrated Design, *Informatics in Education,* 2007, Vol. 6, No. 2, 1–24, Institute of Mathematics and Informatics, Vilnius.

Kynigos, C. (2007b), Half-baked Microworlds in use in Challenging Teacher Educators' Knowing, *international Journal of Computers for Mathematical Learning.* Kluwer Academic Publishers, Netherlands, 12 (2), 87-111.

Lagrange, J-B., Chiappini, G. (2007). Integrating the learning of algebra with technology at the European Level: two examples in the Remath project. *Proceeding of CERME 5.* http://ermeweb.free.fr/CERME5b/, 903-913.

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. New York: Cambridge University Press.

Lester F.K. (Ed.) (2007). *Second Handbook of Research on Mathematics Teaching and Learning*. Information Age Publishing, Inc., Greenwich, Connecticut.

Mitchelmore, M. C. & White, P. (1998). Recognition of angular similarities between familiar physical situations, in A. Olivier & K. Newstead (eds.), *Proceedings of the 20th PME Conference*, Vol. 3, pp. 271–278. Stellenbosch.

Mitchelmore, M. C. & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, N°41, 209–238.

Nicaud, J-F., Bouhineau, D., Chaachoua, H. (2004). Mixing microworld and Cas features in building computer systems that help students learn algebra. *International Journal. of Computer for Mathematical Learning*, 9(2), 169-211.

Niss, M. (2007). Reflections on the State and Trends in Research on Mathematics Teaching and Learning: From Here to Utopia. In, F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning,* 1293-1311. Information Age Publishing, Inc., Greenwich, Connecticut.

Noss, R. & Hoyles, C. (1996) *Windows on Mathematical Meanings*. Kluwer Academic Publishers.

O'Halloran, K. L. (2005). *Mathematical Discourse: Language, Symbolism and Visual images.* London: Continuum.

Papert, S. (1980). *Mindstorms*. New York: Basic Books.

Peirce, C.S. (1931). *Collected Papers*. Harward University Press.

Peirce, C. S. (2003), *Opere*, A cura di Massimo A. Bonfantini, Bompiani, Milano, 2003.

Prediger, S., Arzarello, F., Bosch, M., Lenfant, A. (Eds.) (2008). *Comparing, combining, coordinatins-networking strategies for connecting theoretical approaches. ZDM* 40(2).

Prediger, S., Bikner-Ahsbahs, A., Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM* 40(2), 165-178.

Rabardel P. (1995). *Les hommes et les technologies, approche cognitive des instruments contemporains*. Paris: Armand Colin.

Sensevy, G., Schubauer-Leoni, M.L., Mercier, A., Ligozat, F., Perrot, G. (2005). An Attempt to Model the Teacher's Action in the Mathematics Class. *Educational Studies in Mathematics*, 59 (1-3), 153-181.

Silver, E.A., Herbst, P. (2007). Theory in Mathematics Education Scholarship. In, F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning,* 39-67. Information Age Publishing, Inc., Greenwich, Connecticut.

Sfard, A. (2009). What's all the fuss about gestures? A commentary. *Educational Studies in Mathematics.* 70 (2), 191-200.

Strohecker, C., & Slaughter, A. (2000). Kits for learning and a kit for kitmaking. *CHI '00 Extended Abstracts on Human Factors in Computing Systems*, 149-150.

Tricot A., Plégat-Soutjis F., Camps J.-F. et al. (2003), Utilité, utilisabilité, acceptabilité : interpréter les relations entre trois dimensions de l'évaluation des EIAH. In C. Desmoulins et al. (Eds.), *EIAH 2003*,

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques.* 10 No 2.3, 133–170.

Appendix 1: the six ReMath DDA profiles and the NETLOGO profile

Appendix 2: Cross-case analyses